Modeling Holes and Inclusions by Level Sets in the Extended Finite Element Method

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WEB: http://www.tam.nwu.edu/X-FEM

Outline

1. Extended Finite Element Method (X-FEM)
2. Level Set Method
3. Modeling Holes and Inclusions
4. Partition of Unity Method
5. Governing Equations and Weak Form
6. Discrete Approximation
7. Level Set Function: $\varphi(\mathbf{x}, 0)$
8. Numerical Results
9. Conclusions

Extended Finite Element Method

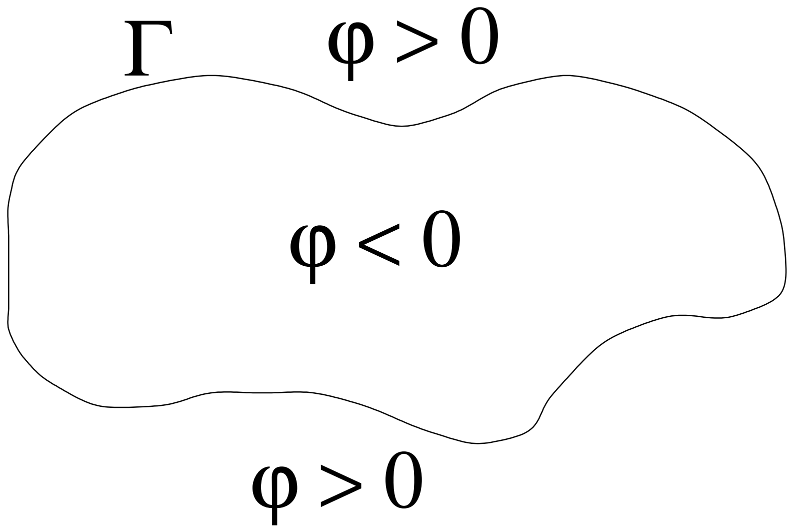
- FE mesh is used to describe the domain
- Geometric discontinuities (cracks) and internal boundaries (holes and material interfaces) are not part of the FE mesh
- Presence of the internal geometric features are ensured by enriching the displacement approximation by additional functions through the notion of partition of unity (PU)

Applications

- Crack modeling in 2D and 3D
- Voids and multiple crack branching
- Mindlin-Reissner plates
- Adaptivity

Level Set Method

- Numerical technique for tracking moving interfaces: interface is represented as the zero level set of a function of one higher dimension: $\varphi(\mathbf{x}, t) = 0$
- Hyperbolic equation in terms of φ governs the motion of the interface
- Key advantages are:
  1. Computed on a fixed Eulerian mesh
  2. Handles topological changes in the interface naturally
  3. Interface properties: $\mathbf{n} = \frac{\boldsymbol{\nabla}\varphi}{||\boldsymbol{\nabla}\varphi||}$, $\kappa = \boldsymbol{\nabla} \cdot \mathbf{n}$
  4. Formulation extends to higher dimensions



Modeling Holes and Inclusions

Finite Elements

- Mesh needs to conform to holes and inclusions
- Since the displacement field in each element is C^0, the interface jump condition is met in a weak sense
- Mesh generation is time-intensive if a wide array of defects need to be analyzed

X-FEM and Level Sets

- Internal boundaries are independent of the mesh
- Additional functions are introduced in the displacement approximation through the notion of PU to model interfaces
- Level set function is used to represent the internal boundaries and to construct the enrichment function for material interfaces
- Single mesh suffices for analysis

Partition of Unity Method

(Melenk and Babuška, 1996)

Addresses the question—How can a function $f(\mathbf{x})$ be introduced in a finite element space over a region $D \subset \Omega$ such that the sparsity (require local approximation) of the stiffness matrix is retained?

Classical FE Approximation in Ω

$$u^h(\mathbf{x}) = \sum_{I}^{n} \phi_I(\mathbf{x}) u_I,$$

where

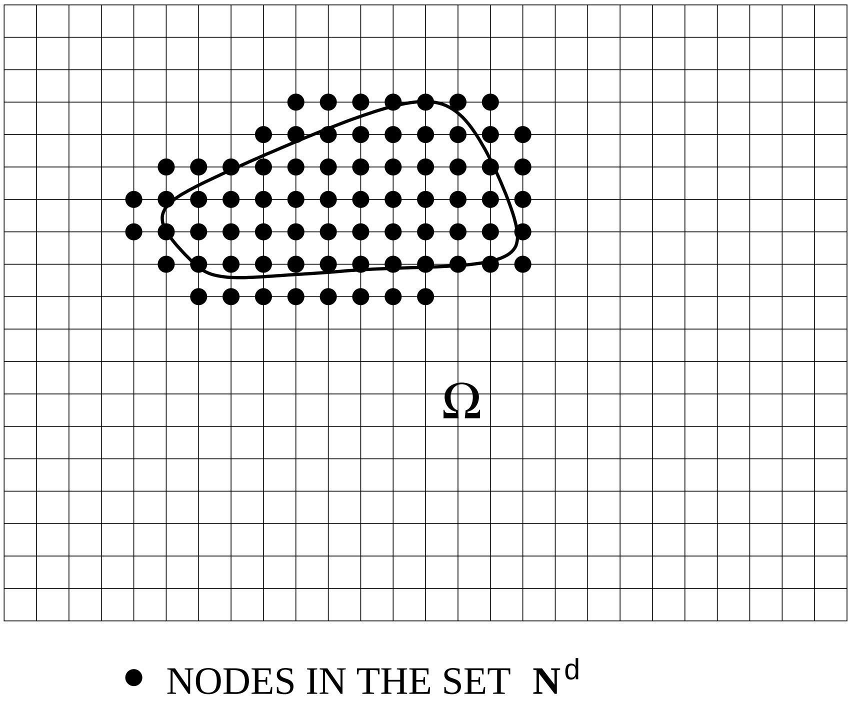
$$\sum_{I}^{n} \phi_I(\mathbf{x}) = 1, \quad \sum_{I}^{n} \phi_I(\mathbf{x}) \mathbf{x}_I = \mathbf{x}.$$

To represent the function $f(\mathbf{x})$ in the sub-region $D \subset \Omega$, an enriched finite element approximation is needed:

$$u^h(\mathbf{x}) = \underbrace{\sum_{\substack{I \\ n_I \in \mathbf{N}}} \phi_I(\mathbf{x}) u_I}_{classical} + \underbrace{\sum_{\substack{J \\ n_J \in \mathbf{N}^d}} \phi_J(\mathbf{x}) a_J f(\mathbf{x})}_{enrichment}$$

$$\mathbf{N}^d = \{n_J : n_J \in \mathbf{N},\ \mathrm{supp}(\phi_J) \cap D \neq \emptyset\}$$

$$\mathbf{N} = \{n_1, n_2, \ldots , n_n\}$$



Consequences

If $u_I = 0$ and $a_I = 1\ \forall\ I$:

$$u^h(\mathbf{x}) = \sum_I \phi_I(\mathbf{x}) f(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in D$$

If $u_I = 0$ and $a_I = x_I$ or $a_I = y_I\ \forall\ I$:

$$u^h(\mathbf{x}) = \sum_I \phi_I(\mathbf{x}) x_I f(\mathbf{x}) = x f(\mathbf{x}), \quad \mathbf{x} \in D$$

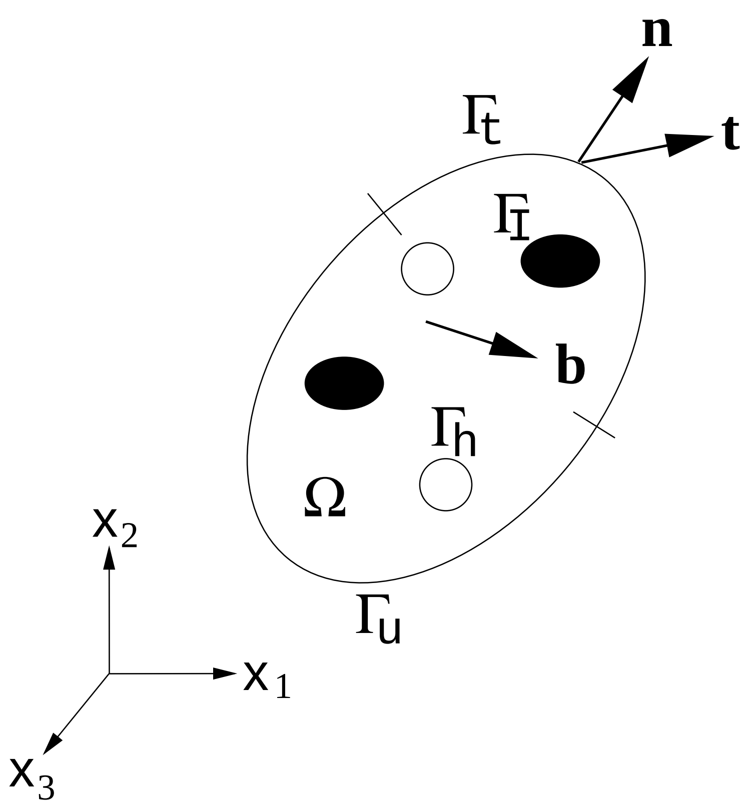
$$u^h(\mathbf{x}) = \sum_I \phi_I(\mathbf{x}) y_I f(\mathbf{x}) = y f(\mathbf{x}), \quad \mathbf{x} \in D$$

Governing Equations of Linear Elasticity

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0 \quad \text{in} \quad \Omega,$$

$$\boldsymbol{\sigma} = \mathrm{C} : \boldsymbol{\varepsilon},$$

$$\boldsymbol{\varepsilon} = \nabla_s \mathbf{u},$$



Boundary Conditions

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \Gamma_u,$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \Gamma_t,$$

$$\boldsymbol{\sigma} \cdot \mathbf{n}_h = 0 \quad \text{on} \quad \Gamma_h,$$

$$[\![\boldsymbol{\sigma} \cdot \mathbf{n}_I]\!] = 0 \quad \text{on} \quad \Gamma_I,$$

$$\Gamma = \Gamma_u \cup \Gamma_t, \;\; \Gamma_u \cap \Gamma_t = \emptyset$$

Variational Statement of BVP

Weak Form

Find $\mathbf{u} \in \mathbf{V}$ such that $\forall\, \mathbf{v} \in \mathbf{V}_0$

$$\int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) = \int_\Omega \mathbf{b} \cdot \mathbf{v}\, d\Omega + \int_{\Gamma_t} \bar{\mathbf{t}} \cdot \mathbf{v}\, d\Gamma$$

$$\mathbf{V} = \left(H^1(\Omega)\right)^2, \ \mathbf{V}_0 = \left(H^1_0(\Omega)\right)^2$$

Weak Form (Discrete)

Find $\mathbf{u}^h \in \mathbf{V}^h \subset \mathbf{V}$ such that $\forall\, \mathbf{v}^h \in \mathbf{V}^h_0 \subset \mathbf{V}_0$

$$\int_{\Omega^h} \boldsymbol{\sigma}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\mathbf{v}^h) = \int_{\Omega^h} \mathbf{b} \cdot \mathbf{v}^h\, d\Omega + \int_{\Gamma^h_t} \bar{\mathbf{t}} \cdot \mathbf{v}^h\, d\Gamma$$

Remark: For a crack, $\mathbf{V}^h$ must contain a discontinuous field, whereas for a material interface Γ_I with outward normal $\mathbf{n}$, $\mathbf{V}^h$ must contain a function f such that $[\![\boldsymbol{\nabla} f \cdot \mathbf{n}]\!] \neq 0$ across Γ_I.

Discrete Approximation

Trial and Test Functions

$$\mathbf{u}^h(\mathbf{x}) = \sum_I \phi_I(\mathbf{x})\mathbf{u}_I + \sum_J \phi_J(\mathbf{x})\psi\big(\varphi(\mathbf{x})\big)\mathbf{a}_J,$$

$$\mathbf{v}^h(\mathbf{x}) = \sum_I \phi_I(\mathbf{x})\mathbf{v}_I + \sum_J \phi_J(\mathbf{x})\psi\big(\varphi(\mathbf{x})\big)\mathbf{b}_J,$$

$\phi_I(\mathbf{x})$: FE shape function,

$\varphi(\mathbf{x})$: Level set function,

$\psi\big(\varphi(\mathbf{x})\big)$: Enrichment function for material interface

Discrete System

$$\mathbf{K}\mathbf{d} = \mathbf{f},$$

$$\mathbf{K}_{IJ} = \int_{\Omega^h} \mathbf{B}^T\mathbf{C}\mathbf{B}\, d\Omega,$$

$$\mathbf{f}_I = \int_{\Gamma_t^h} \hat{\phi}_I\bar{\mathbf{t}}\, d\Gamma + \int_{\Omega^h} \hat{\phi}_I\mathbf{b}\, d\Omega,$$

$$\text{FE DOF} : \hat{\phi}_I = \phi_I, \ \text{Enriched DOF} : \hat{\phi}_I = \phi_I\psi$$

Level Set Function: $\varphi(\mathbf{x}, 0)$

Circular

$$\varphi_I = \min_{\substack{\mathbf{x}_c^i \in \Omega_c^i \\ i=1,2,\ldots,n_c}} \left\{ \|\mathbf{x}_I - \mathbf{x}_c^i\| - r_c^i \right\}$$

Elliptical

$$\varphi_I = \min_{i=1,2,\ldots,n_e} f(\boldsymbol{\xi}^i),$$

$$f(\boldsymbol{\xi}^i) = \|\boldsymbol{\xi}^i\| - 1,$$

$$\boldsymbol{\xi}^i = \left(\frac{\hat{x}_1^i}{a_i}, \frac{\hat{x}_2^i}{b_i} \right),$$

$$\hat{\mathbf{x}}^i = \mathbf{R}^i (\mathbf{x}_I - \mathbf{x}_c^i)$$

Polygonal

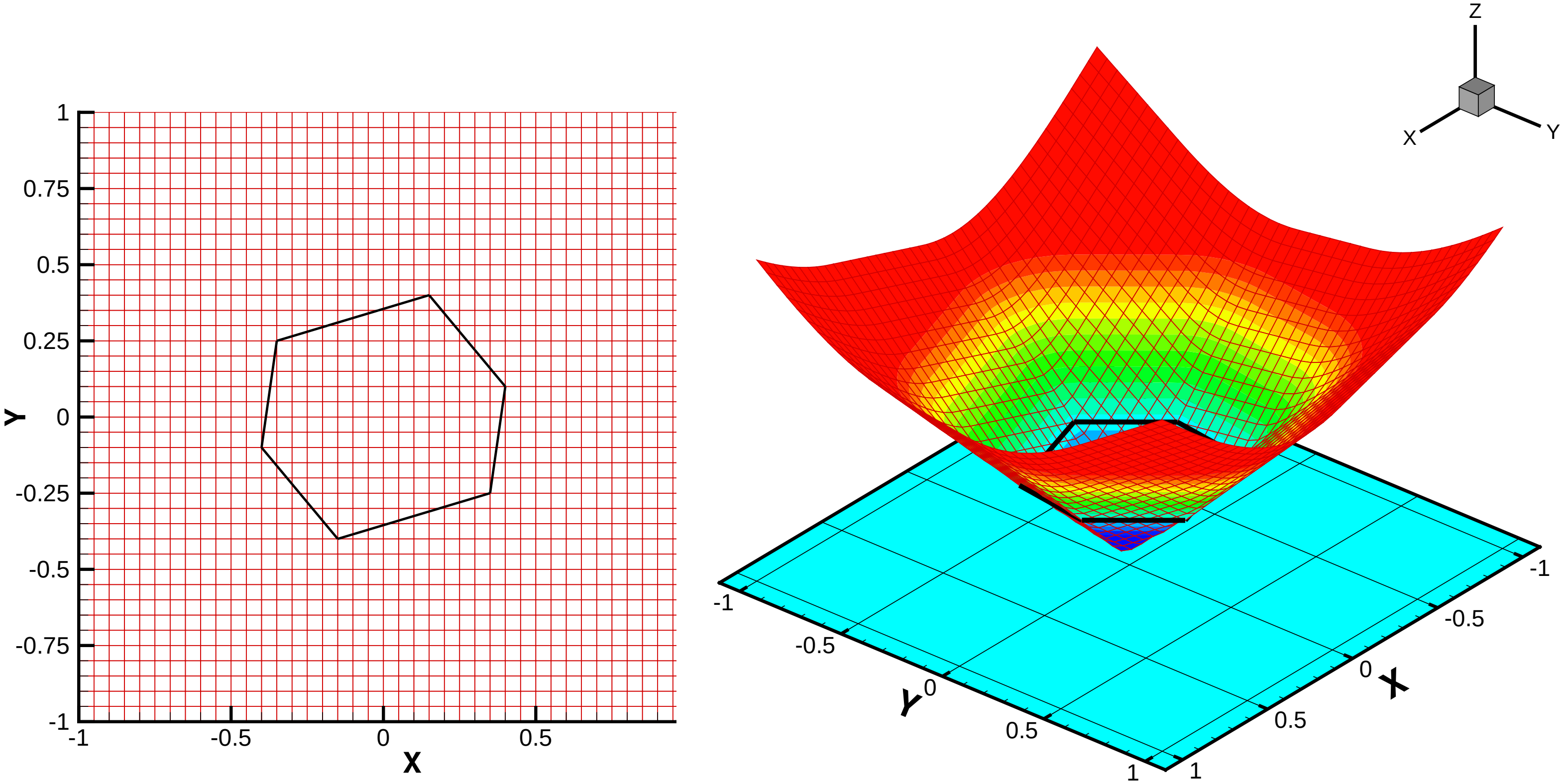
$$\Gamma_p = \cup_{i=1}^p \mathbf{l}_i, \;\; \mathbf{l}_1 = [\mathbf{x}_1, \, \mathbf{x}_2], \ldots, \mathbf{l}_p = [\mathbf{x}_p, \, \mathbf{x}_1],$$

$$\varphi_I = \|\mathbf{x}_I - \mathbf{x}_{\min}\| \; \mathrm{sgn}\big((\mathbf{x}_I - \mathbf{x}_{\min}) \cdot \mathbf{n}_{\min} \big),$$

$$\mathrm{sgn}(\xi) = (\xi \geq 0) \; ? \; 1 : -1,$$

$$\|\mathbf{x}_I - \mathbf{x}_{\min}\| = \min_{\substack{\mathbf{x}_i \in \mathbf{l}_i \\ i=1,2,\ldots,p}} \|\mathbf{x}_I - \mathbf{x}_i\|$$

$$\boxed{\text{Interpolation: } \varphi(\mathbf{x}) \equiv \varphi(\mathbf{x}, 0) = \sum_I \phi_I(\mathbf{x}) \varphi_I}$$

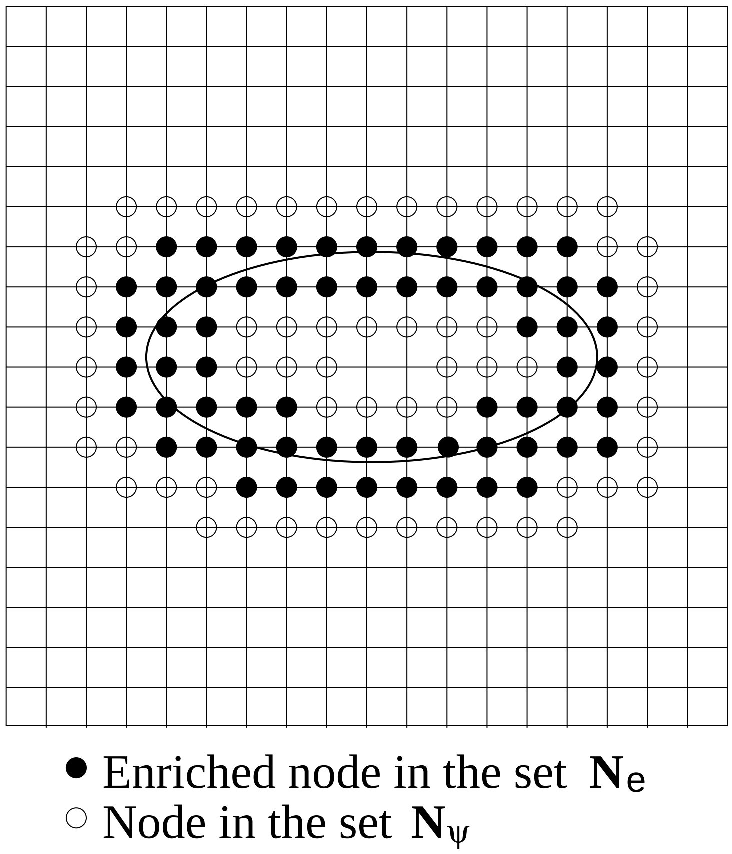


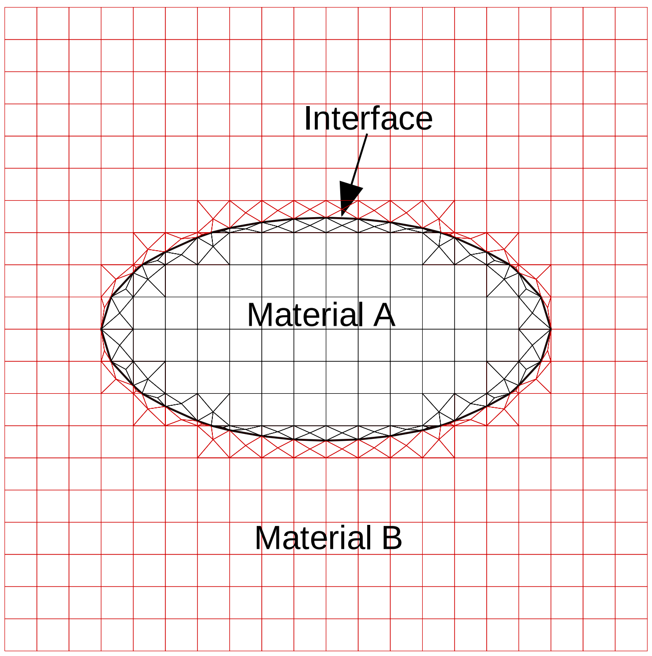
Level set function for a hexagonal interface

Nodal Enrichment and Partitioning

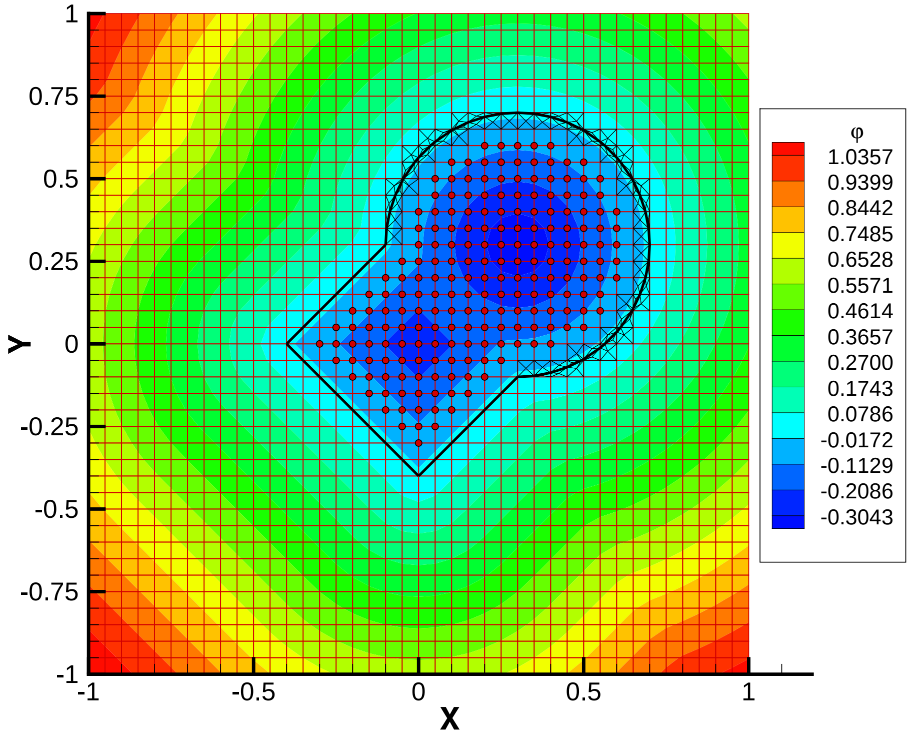
Algorithm

- Initialize: $\mathbf{T} = \{\}$
- for $i = 1$ to m ($t_i = i$th element)
  - if $\varphi_I \varphi_J < 0$ for $(n_I, n_J) \in t_i$ $\{\, \mathbf{T} \leftarrow \{\mathbf{T}, t_i\} \,\}$
- end for
- $\mathbf{N}_e = \{n_I : n_I \in t_i,\ t_i \in \mathbf{T}\}$
- $\mathbf{N}_\psi = \{n_I : n_I \notin \mathbf{N}_e, n_J \in \mathbf{N}_e,\ (n_I, n_J) \in t_k\}$
- for $i = 1$ to m_c ($m_c = \texttt{Cardinality}(\mathbf{T})$, $t_i \in \mathbf{T}$)
  - e_i = edge of t_i
  - if $\varphi_I \varphi_J < 0$ for $(n_I, n_J) \in e_i$
  - find intersection points $\mathbf{a}$ and $\mathbf{b}$:
    $$\mathbf{x}_p = \mathbf{x}_I + \xi(\mathbf{x}_J - \mathbf{x}_I),\ \xi = -\frac{\varphi_I}{\varphi_J - \varphi_I}$$
  - end if
  - partition t_i
- end for





Enriched nodes and material typing


φ
1.0357
0.9399
0.8442
0.7485
0.6528
0.5571
0.4614
0.3657
0.2700
0.1743
0.0786
-0.0172
-0.1129
-0.2086
-0.3043
Y
X
1
0.75
0.5
0.25
0
-0.25
-0.5
-0.75
-1
-0.5
0
0.5
1

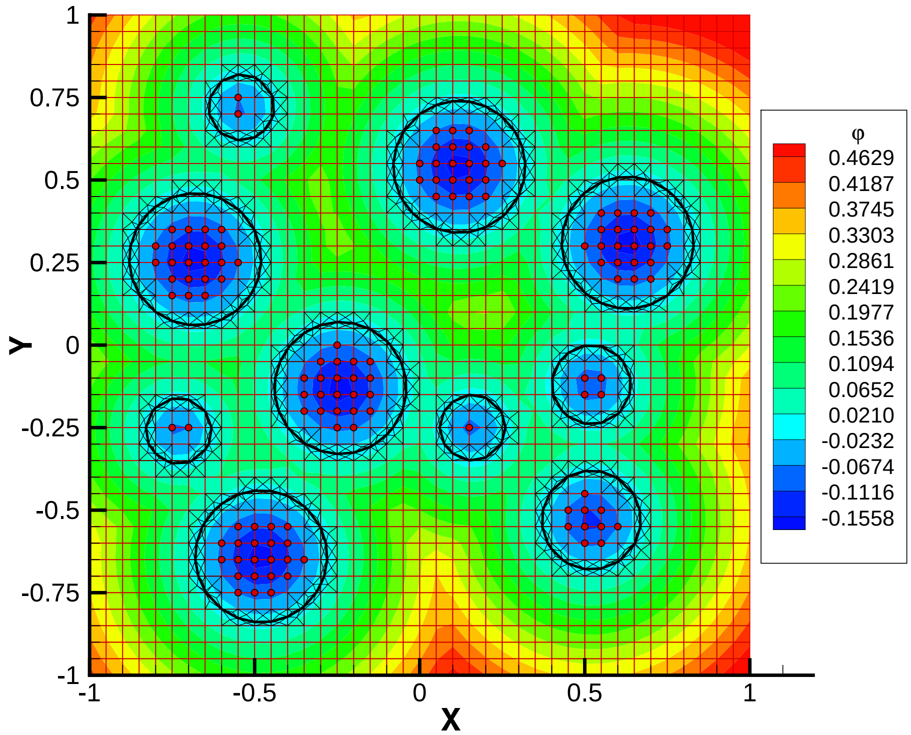
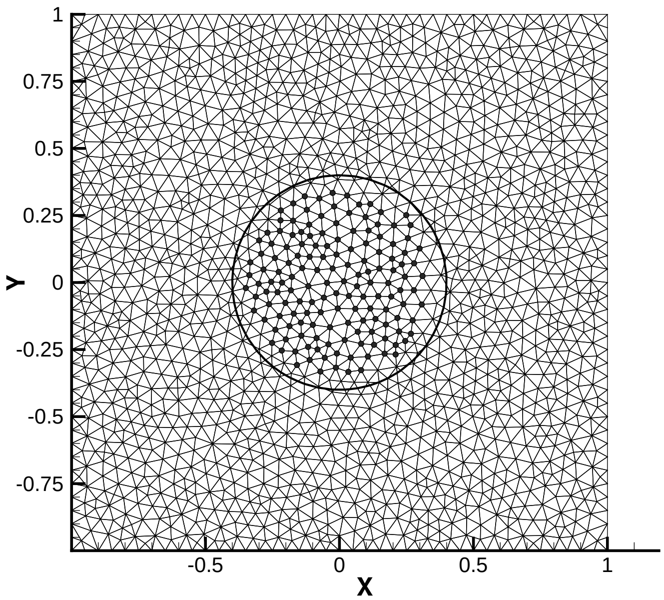
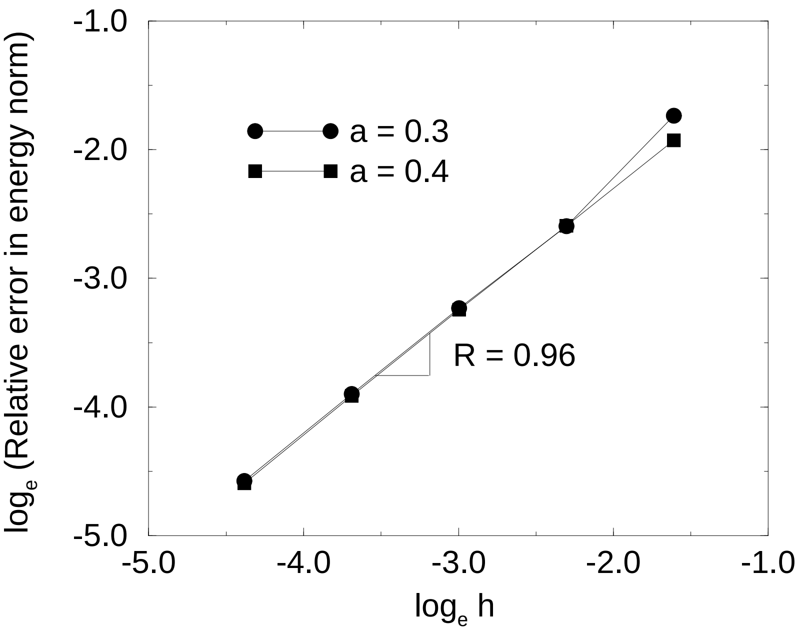

φ
0.4629
0.4187
0.3745
0.3303
0.2861
0.2419
0.1977
0.1536
0.1094
0.0652
0.0210
-0.0232
-0.0674
-0.1116
-0.1558
Y
X
1
0.75
0.5
0.25
0
-0.25
-0.5
-0.75
-1
-0.5
0
0.5
1

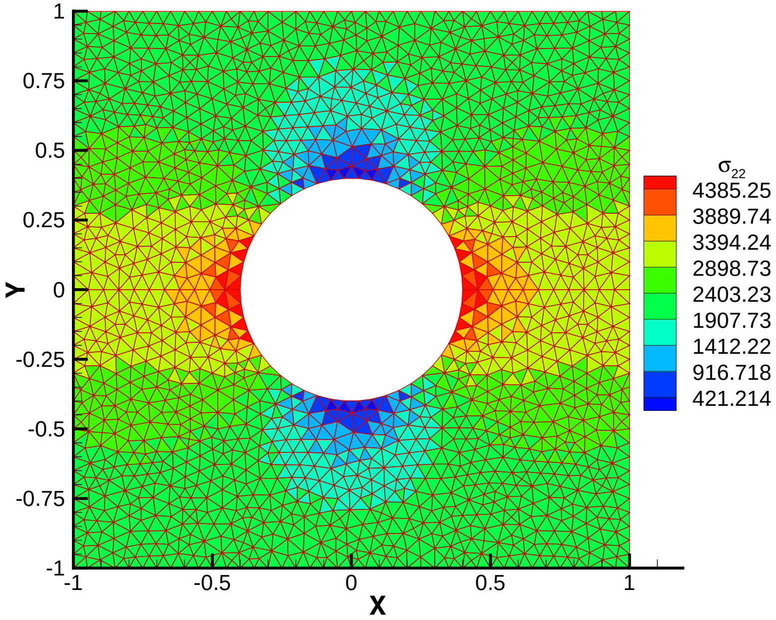
Plate with a Hole



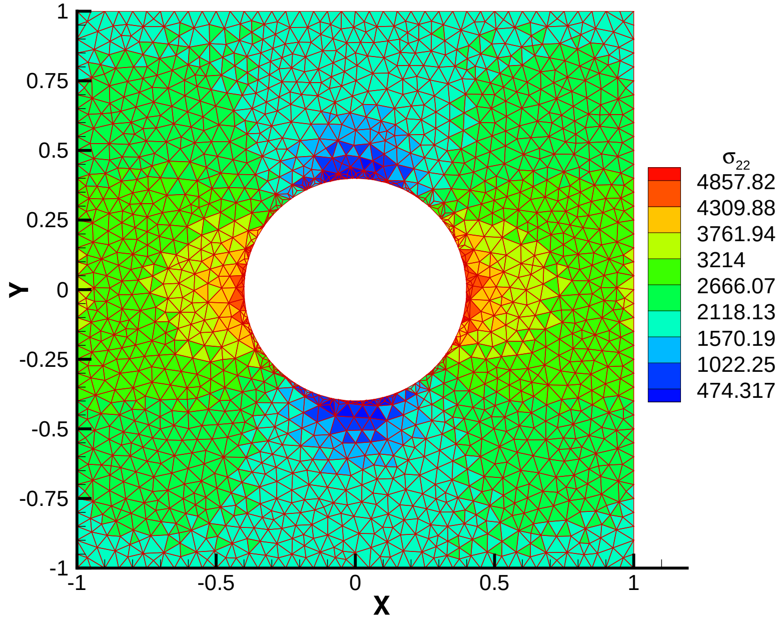
Mesh



Rate of convergence in energy



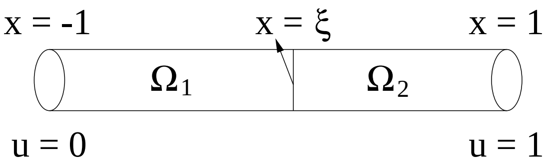
(a) FE



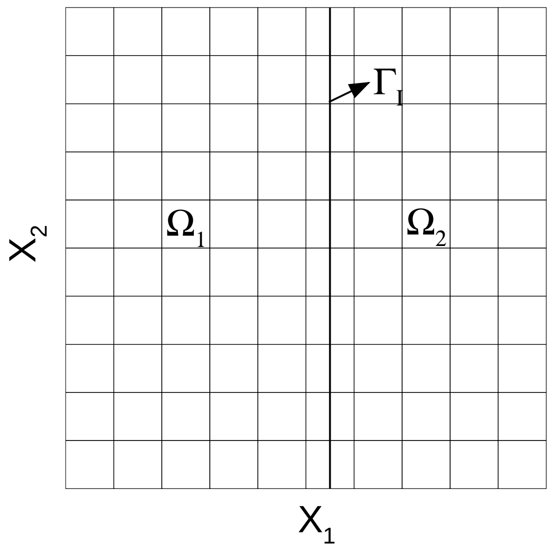
(b) X-FEM

Contour plot of the normal stress σ_{22}

Patch Test



1D bar



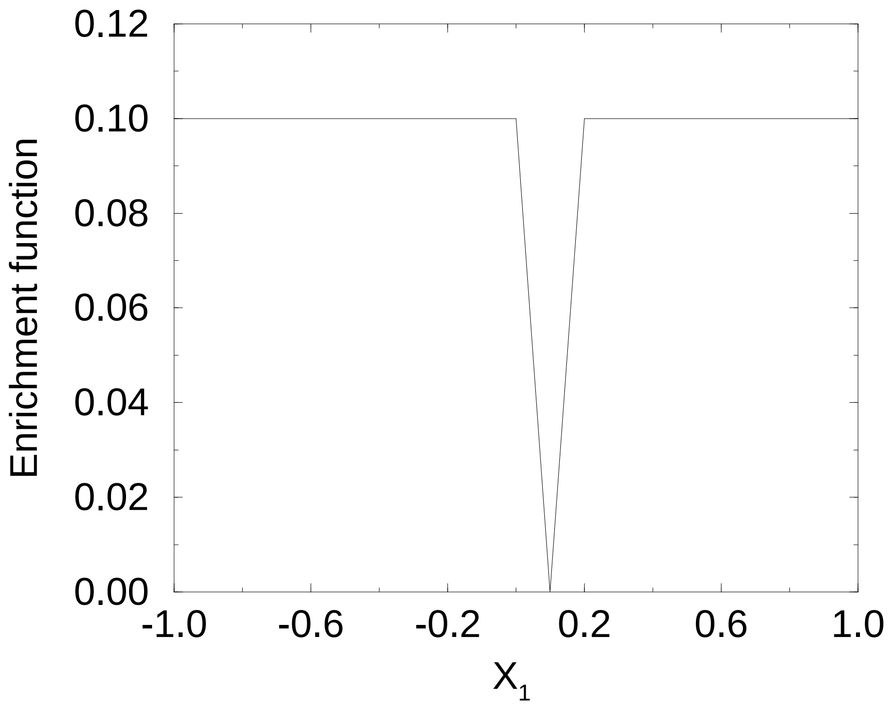
Mesh

Exact Solution

$$u_1(x) = \begin{cases} (1+x_1)\alpha & -1 \le x_1 \le \xi \\ 1 + \dfrac{E_1}{E_2}(x_1 - 1)\alpha & \xi \le x_1 \le 1 \end{cases},$$

where

$$\alpha = \frac{E_2}{E_2(1+\xi) - E_1(\xi - 1)}.$$



Enrichment function $\hat{\psi}$ $(\xi = 0.1)$

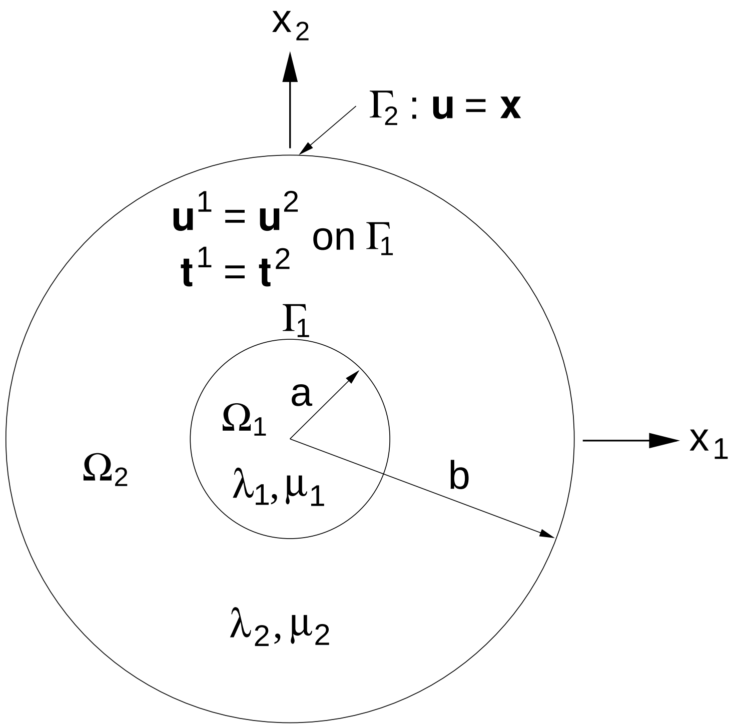
Relative energy norm

ξ	$\lvert\varphi\rvert$	$\hat{\psi}$
0.01	8.3×10^{-2}	3.0×10^{-8}
0.05	1.6×10^{-1}	2.8×10^{-8}
0.10	1.8×10^{-1}	2.1×10^{-8}
0.15	1.8×10^{-1}	3.8×10^{-8}
0.19	1.6×10^{-1}	3.6×10^{-8}

Enrichment Functions for Interfaces

1. $\psi(\mathbf{x}) = |\varphi(\mathbf{x})|$
2. $\psi(\mathbf{x}) = \psi_1(\mathbf{x})$ (smoothing procedure)
   - recall enriched nodal set $\mathbf{N}_e$ and the set $\mathbf{N}_\psi$
   - let $\mathbf{N}_\psi = \{n_1, n_2, \ldots, n_p\}$ be sorted in $\uparrow \varphi_I$
   - set $\psi_I = \varphi_I$ for $n_I \in \mathbf{N}_{e\psi}$, $\mathbf{N}_{e\psi} = \mathbf{N}_e \cup \mathbf{N}_\psi$
   - New values of ψ_I $(n_I \in \mathbf{N}_\psi)$ are computed:
     - for $J = 1$ to p
       (a) create the set $\mathbf{P}_J = \{n_I \in \mathbf{N}_{e\psi} :$
       $|\psi_I| < |\varphi_J|, \; (n_I, n_J) \in t_k\}$
       (b) solve: $\min \sum_{n_I \in \mathbf{P}_J} \left(\frac{\psi_J - \varphi_I}{L_I} \right)^2$ for ψ_J,
       $L_I = d(\mathbf{x}_I, \mathbf{x}_J)$
       (c) $\psi_J = \sum_{n_I \in \mathbf{P}_J} \alpha_I \varphi_I, \; \alpha_I = \dfrac{\frac{1}{L_I^2}}{\sum_{n_K \in \mathbf{P}_J} \frac{1}{L_K^2}}$
     - end for
   - $\psi(\mathbf{x}) \equiv \psi_1(\mathbf{x}) = \left| \sum_I \phi_I(\mathbf{x}) \psi_I \right|$

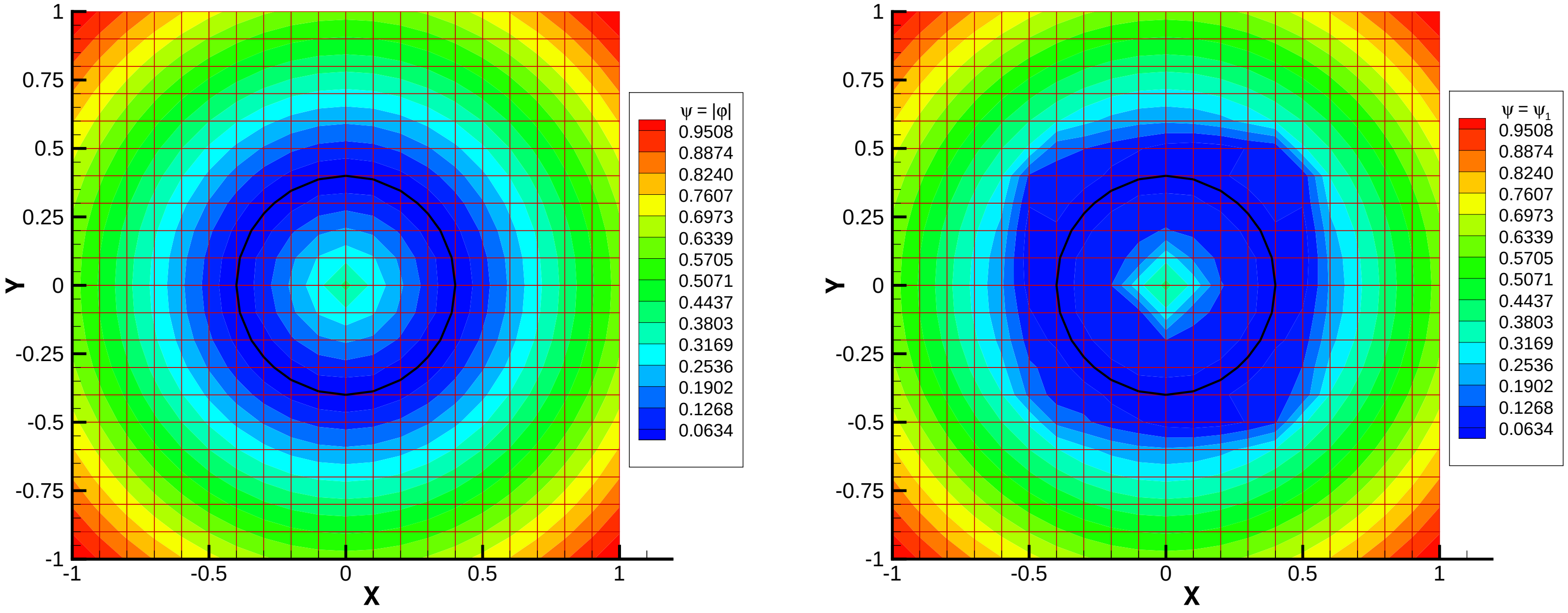
Bimaterial BVP



$$\text{Navier's equation: } \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru_r)\right] = 0$$

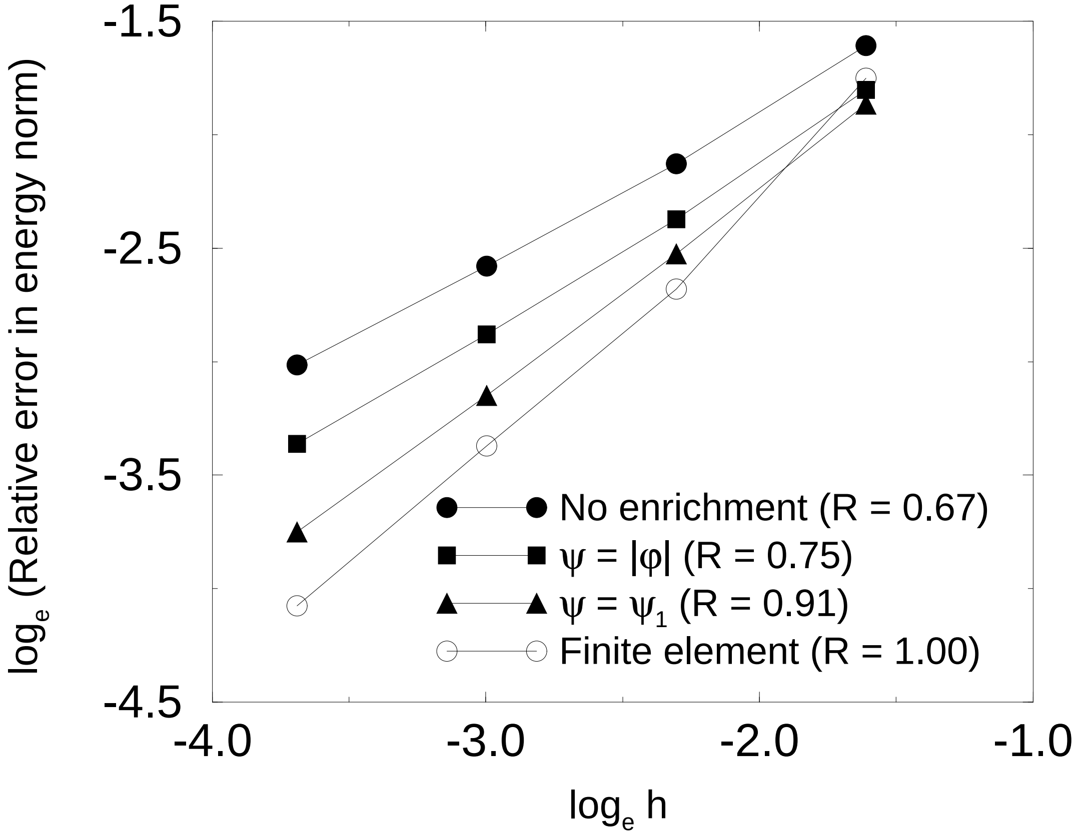
$$u_r(r) = \begin{cases} \left[\left(1 - \dfrac{b^2}{a^2}\right)\alpha + \dfrac{b^2}{a^2}\right] r, & 0 \le r \le a \\ \left(r - \dfrac{b^2}{r}\right)\alpha + \dfrac{b^2}{r} & a \le r \le b \end{cases},$$

$$\alpha = \frac{(\lambda_1 + \mu_1 + \mu_2)b^2}{(\lambda_2 + \mu_2)a^2 + (\lambda_1 + \mu_1)(b^2 - a^2) + \mu_2 b^2}$$



(a) $|\varphi|$ (b) ψ_1

Enrichment functions for material interfaces



Rate of convergence in energy

Conclusions

1. A numerical technique that couples the X-FEM and the levet set method was proposed to model arbitrary holes and material interfaces without meshing the internal boundaries

2. The level set function φ is used to represent holes and inclusions, and φ is also used to develop the local enrichment for interfaces

3. Issues pertaining to the appropriate choice of enrichment functions for material interfaces were addressed, and a suitable enrichment function was derived that yielded near optimal rate of convergence for a bimaterial BVP

4. The methodology presented in this study is of merit in the modeling of arbitrary defects in multi-phase materials