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CRACK ANALYSIS IN MAGNETOELECTROELASTIC MEDIA USING THE EXTENDED FINITE ELEMENT METHOD

RAMÓN ROJAS-DÍAZ*, NATARAJAN SUKUMAR†, ANDRÉS SÁEZ* AND FELIPE GARCÍA-SÁNCHEZ††

*Departamento de Mecánica de los Medios Continuos, Escuela Técnica Superior de Ingenieros, Universidad de Sevilla, 41092-Sevilla, Spain.

e-mail: rrojasdiaz@us.es, andres@us.es

†Department of Civil and Environmental Engineering, University of California, Davis, CA 95616, USA.

e-mail: nsukumar@ucdavis.edu

††Departamento de Ingeniera Civil, de Materiales y Fabricación, ETS de Ingenieros Industriales, 29013 Málaga

e-mail: fgsanchez@uma.es

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Summary. An Extended Finite Element Method (X-FEM) for the study of fracture in magnetoelectroelastic solids is presented. The extended stress intensity factors are extracted using the domain form of the interaction integral. Good agreement of the extended £nite element solution with analytical and boundary element solutions is demonstrated.

1 INTRODUCTION

Magnetoelectroelastic materials are receiving increasing attention due to the potential they have in innovative applications in smart or intelligent devices and structures. These materials present full coupling between mechanical, electric and magnetic £elds [1]. These materials present a big tendency to develop cracks due to their inner fragility and the fabrication process to obtain them, which provides motivation to study fracture in magnetoelectroelastic media. This paper presents crack analysis of linear magneto-electroelastic materials subjected to different static combinations of mechanical, electric and magnetic conditions. To this end, an ef£cient extended £nite element (X-FEM) formulation is introduced. There exist previous studies that have revealed the accuracy of the method for isotropic [2, 3], orthotropic [4], and piezoelectric solids [5]. In the present work, the enrichment functions for magnetoelectroelastic materials together with an appropriate expression for the interaction integral are developed. We demonstrate the excellent accuracy of the X-FEM by comparing the numerical results to those obtained by the authors using a hypersingular formulation of the boundary element method [6].

2 BASIC EQUATIONS OF LINEAR MAGNETOELECTROELASTICITY

A coupling between electric and magnetic effects and the elastic £elds appear in magnetoelectroelastic materials. Constitutive equations relate mechanical stresses σ_{ij}, electric displacements D and magnetic induction B with the elastic strains ϵ_{ij} and the electric and magnetic £elds E_i and H_i. These equations are given by

$$\begin{aligned} \sigma_{ij} &= c_{ijkl}\varepsilon_{kl} - e_{lij}E_l - h_{lij}H_l \\ D_i &= e_{ikl}\varepsilon_{kl} + \epsilon_{il}E_l + \beta_{il}H_l \\ B_i &= h_{ikl}\varepsilon_{kl} + \beta_{il}E_l + \gamma_{il}H_l, \end{aligned} \tag{1}$$

where C_{ijkl}, ϵ_{il} and γ_{il} are the elastic stiffness tensor, dielectric permittivities and magnetic permeabilities, respectively, and e_{lij}, h_{lij} and β_{il} are the piezoelectric, piezomagnetic and electromagnetic coupling coef£cients, respectively. Then, a generalized stiffness tensor can be de£ned as

$$C_{iJKl} = \begin{cases} Cijkl, & \text{J,K=1,2,3} \\ e_{ijl}, & \text{J=1,2,3; K=4} \\ h_{ijl}, & \text{J=1,2,3; K=5} \\ -\epsilon_{il}, & \text{J=4} \\ -\beta_{il}, & \text{J=4; K=5} \\ -\gamma_{il}, & \text{J=K=5,} \end{cases} . \tag{2}$$

The linear magnetoelectroelastic problem may be formulated in an elastic-like fashion by considering a generalized displacement vector and stress tensor. The displacement vector is extended with the electric and magnetic potentials, while the tress tensor is extended by the electric displacement and the magnetic induction:

$$u_I = \begin{cases} u_i, & \text{J=1,2,3} \\ \phi_i, & \text{J=4} \\ \varphi_i, & \text{J=5,} \end{cases} \tag{3}$$

$$\sigma_{iJ} = \begin{cases} \sigma_{ij}, & \text{J=1,2,3} \\ D_i, & \text{J=4} \\ B_i, & \text{J=5,} \end{cases} \tag{4}$$

Using the extended variables, the equilibrium equations can be expressed as

$$\sigma_{iJ,i} + b_J = 0, \tag{5}$$

where

$$\sigma_{iJ} = C_{iJKl}u_{K,l} = 0. \tag{6}$$

3 EXTENDED FINITE ELEMENT METHOD FORMULATION

The extended £nite element method [2, 3] is a technique to simulate crack discontinuities without the need for the crack to conform to the £nite element mesh. To this end, additional (enrichment) functions are added to the classical £nite element approximation through the framework of partition of unity [7]. The crack interior is represented by a discontinuous (Heaviside) function and the crack-tip is modeled by the asymptotic crack-tip functions.

3.1 Crack modelling and selection of enriched nodes

Let us consider a arbitrary domain with a single crack discontinuity in it. The domain is discretized into elements; we denote the nodal set as $\mathcal{N}$. Then, the displacement approximation (trial function) is written as [3]

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I \in N} N_I(\mathbf{x})\mathbf{u}_I + \sum_{J \in N^H} N_J(\mathbf{x})H(\mathbf{x})\boldsymbol{a}_J + \sum_{K \in N^{CT}} N_K(\mathbf{x}) \sum_l F_l(\mathbf{x})\boldsymbol{b}^l_K, \tag{7}$$

where N_I is the shape function associated with the node I, $\mathbf{u}_I$ is the vector of the traditional nodal degrees of freedom de£ned in in £nite elements discretization, and $\mathbf{a}_J$ and $\mathbf{b}^l_K$ are the additional enriched degrees of freedom. In the above equation, $H(\mathbf{x})$ is the generalized Heaviside function that simulates the displacement discontinuity on both sides of the crack faces, and F_l are the crack tip enrichment functions.

In a magnetoelectroelatic solid, the variables which appear in (7) are de£ned in an extended way, so **u** and **a** are 4 component-vectors and **b** is a 32 component-vector.

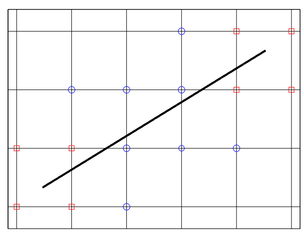

Figure 1: Node selection for enrichment

In a £nite element mesh, as seen in £gure (1), the set of nodes that are enriched with Heaviside functions ($\mathcal{N}^H$) are marked with a circle, whereas the set of nodes that are enriched with crack tip enrichment functions ($\mathcal{N}^{CT}$) are marked with a square.

3.2 Enrichment functions

In an extended £nite element formulation, the near-tip asymptotic functions are used to model the crack tip. These functions, known as crack tip enrichment functions, must span all the possible displacement £elds around the crack tip, for any orientation of the crack and loading combination.

For magnetoelectroelastic materials eight functions are needed to describe all the possible displacement states around the crack tip, while for isotropic and piezoelectric materials, only four or six functions are needed. These functions, named as F_l in equation (7), are obtained from the asymptotic singular solution and can be expressed as an extension of those obtained in Reference [5] for piezoelectric materials. The details on the derivation of the crack-tip enrichment functions are provided in Reference [8].

$$\begin{aligned} F_l(r,\theta) = \sqrt(r)\{ \ \rho_1\cos(\theta_1/2) \quad \rho_2\cos(\theta_2/2) \quad \rho_3\cos(\theta_3/2)) \quad \rho_4\cos(\theta_4/2) \\ \rho_1\sin(\theta_1/2) \quad \rho_2\sin(\theta_2/2) \quad \rho_3\sin(\theta_3/2) \quad \rho_4\sin(\theta_4/2) \ \} \end{aligned} \tag{8}$$

where

$$\rho_K(\omega,\mu_I) = \frac{1}{\sqrt{2}}\sqrt[4]{|\mu_K|^2 + \Re(\mu_K)\sin 2\omega - [|\mu_K|^2 - 1]\cos 2\omega} \tag{9}$$

$$\theta_K = \pi \text{Int}(\frac{\omega}{\pi}) + \arctan \frac{|\Im(\mu_K)| \sin(\omega - \pi \text{Int}(\frac{\omega}{\pi}))}{\cos(\omega - \pi \text{Int})(\frac{\omega}{\pi}) + |\Re(\mu_K)| \sin(\omega - \pi \text{Int}(\frac{\omega}{\pi}))}, \quad (10)$$

where $\omega = \theta - \alpha$, and α is the orientation angle of the material axes with respect to the crack and μ_I are the four roots of the characteristic equation with positive imaginary part.

4 Computation of fracture parameters

Stress intensity facto are used to to quantify the fracture process in the vicinity of a crack tip. Since in magnetoelectroelastic materials, the displacement vector is de£ned in an extended way, and the extended crack opening displacements have also a $\sqrt{r}$ behaviour, the stress intensity factors are now de£ned in an extended way. In this paper, we use the technique developed by Rao and Kuna [9] for extracting SIFs in magnetoelectroelastic materials using the interaction integral method.

5 NUMERICAL RESULTS

Two static crack problems in magnetoelectroelastic media are solved by the use of the X-FEM. The numerical results obtained are compared with those obtained by the boundary element formulation presented in Reference [6]. In all simulations, a magnetoelectroelastic composite material $BaTiO_3$-$CoFe_2O_4$ with a volume fraction $V_f = 0.5$ is considered, whose properties can be found in Table 1.

Table 1: Material properties of $BaTiO_3$–$CoFe_2O_4$ (V_f=0.5).

C_{11} (GPa) 226	C_{12} (GPa) 125	C_{13} (GPa) 124	C_{33} (GPa) 216	$C_{44}(GPa)$ 44
e_{31} (C/m^2) -2.2	e_{33} (C/m^2) 9.3	e_{15} (C/m^2) 5.8		
h_{31} (N/Am) 290.2	h_{33} (N/Am) 350	h_{15} (N/Am) 275		
ϵ_{11} (C^2/Nm^2) 56.4×10^{-10}	ϵ_{33} (C^2/Nm^2) 63.5×10^{-10}			
β_{11} (Ns/VC) 5.367×10^{-12}	β_{33} (Ns/VC) 2737.5×10^{-12}			
γ_{11} (Ns^2/C^2) 297×10^{-6}	γ_{33} (Ns^2/C^2) 83.5×10^{-6}			

In all simulations, 2×2 Gauss quadrature is used in every non-enriched element, whereas for a non-partitioned enriched element a 5×5 Gauss rule is used. For enriched elements that are partitioned into subtriangles, a seven point Gauss rule is used in each subtriangle. Moreover, linear quadrilateral elements have been used in all numerical experiments.

5.1 Slanted crack in a magnetoelectroelastic plate

A £nite magnetoelectroelastic plate with a central inclined crack under combined electro-magneto-mechanical loads (see Figure 2) is analyzed for different angles of the crack respect of the horizontal. The ratio between the crack length and plate width is $a/w = 0.2$. The plate is under uniform tension in the

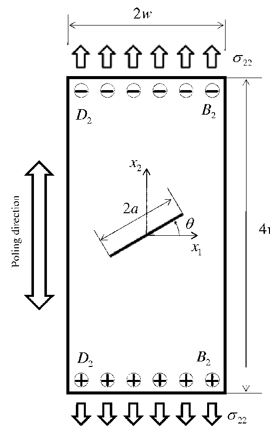

Figure 2: Inclined crack in a rectangular magnetoelectroelastic plate

x_2 direction and subjected to both electric and magnetic loading: $D_2 = 0.1 \cdot 10^{-9}\sigma_{22}CN^{-1}$ and $B_2 = 1 \cdot 10^{-9}\sigma_{22}A^{-1}m$. The problem has been solved with three different uniform meshes. The extended £nite element solutions are compared with those obtained with the hypersingular BEM formulation developed by García-Sáchez *et al.* [6]. The polarization direction considered coincides with the x_2 direction.

Table 2: Extended SIFs for a crack in a £nite plate.

α	$ESIF's$	(25 x 50)	(50 x 100)	(75 x 150)	$K_J^{BEM}/(\sigma_{2J}\sqrt{\pi a})$
0^o	K_I/K_I^{BEM}	0.9822	0.9911	0.9916	1.0241
	K_{II}/K_{II}^{BEM}	~ 1	~ 1	~ 1	~ 0
	K_{IV}/K_{IV}^{BEM}	0.9901	0.9940	0.9952	1.0226
	K_V/K_V^{BEM}	0.9561	0.9827	0.9846	1.0395
15^o	K_I/K_I^{BEM}	1.0256	0.9918	0.9951	0.9562
	K_{II}/K_{II}^{BEM}	1.0311	0.9885	0.9876	0.2506
	K_{IV}/K_{IV}^{BEM}	1.0359	1.0181	1.0186	0.9869
	K_V/K_V^{BEM}	0.9718	0.9575	0.9723	1.0103
30^o	K_I/K_I^{BEM}	0.9803	1.0062	1.0116	0.7720
	K_{II}/K_{II}^{BEM}	1.0541	1.0071	0.9998	0.4361
	K_{IV}/K_{IV}^{BEM}	1.0372	1.0178	1.0137	0.8845
	K_V/K_V^{BEM}	0.9472	0.9995	0.9987	0.9206

In Table 2 the results obtained via X-FEM are normalized by those obtained by the use of BEM as well as the adimensional values of the ESIF's are shown. Good agreement between both formulations is realized.

6 CONCLUSIONS

In this paper, we developed an extended £nite element formulation for fracture problems in bidimensional magnetoelectroelastic media. New crack tip enriching functions were obtained, following the method used by Béchet *et al.* [5] for piezoelectric solids. The robustness of the X-FEM for magnetoelectroelastic fracture problems was demonstrated by evaluating extended stress intensity factors by means of the interaction integral method and comparing them with the results obtained by the boundary element formulation developed by the authors in Reference [6].

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