SIF Computations for 2D Cracks using Enriched Basis Functions (Coupled EFG)

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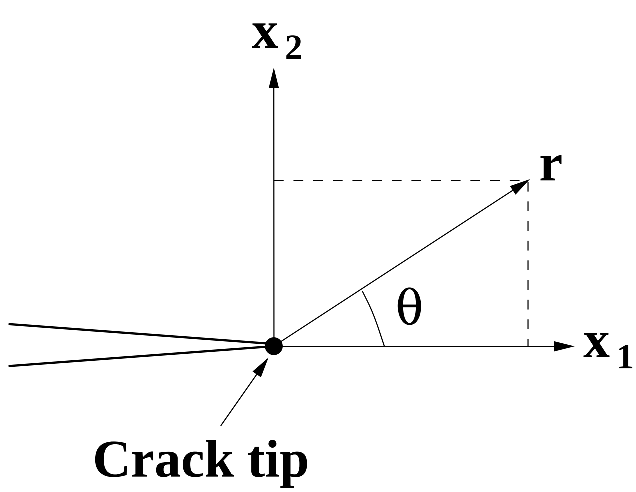
1 Introduction

In this report, stress intensity factor (SIF) computations using the $2D$ coupled FE-EFG program are presented. Two benchmark problems are considered: (1) Edge-cracked plate under tension, and (2) Edge-cracked plate under shear. In order to capture the singular stress fields at the crack-tip, the functions appearing in the asymptotic expansion of the displacements u_i are incorporated into the basis function $\mathbf{p}(\mathbf{x})$.



The **enriched basis function** $\mathbf{p}(\mathbf{x})$ is given by

$$\boldsymbol{p}^T(\boldsymbol{x}) = [\ 1,\ x_1,\ x_2,\ \sqrt{r}\cos\frac{\theta}{2},\ \sqrt{r}\sin\frac{\theta}{2},\ \sqrt{r}\sin\theta\sin\frac{\theta}{2},\ \sqrt{r}\sin\theta\cos\frac{\theta}{2}\ ]. \tag{1}$$

2 EFG Parameters and SIF Computations

The material constants are: $E = 30 \times 10^6$ psi and $\nu = 0.25$. Plane strain conditions are assumed in the analysis. A quartic polynomial weight function is chosen:

$$w_I(\mathbf{x}) = \begin{cases} 1 - 6\left(\frac{d_I}{d_{mI}}\right)^2 + 8\left(\frac{d_I}{d_{mI}}\right)^3 - 3\left(\frac{d_I}{d_{mI}}\right)^4 & \text{if } d_I < d_{mI} \\ \\ 0 & \text{if } d_I \geq d_{mI} \end{cases}$$

where

$$d_I = \|\mathbf{x} - \mathbf{x}_I\|\,,\ d_{mI} = r_{dm}\,c\,,\ c = \alpha c_I\,. \tag{2}$$

The parameter c_I is taken as the distance to the third nearest neighbor. The constant r_{dm} is set to $2\sqrt{2}$ and the dilation parameter $\alpha = 1.001$. The shape functions are computed using the visibility criterion. The solution of the linear system of equations is carried out using a conjugate gradient linear equation solver with block Jacobi preconditioner (***PETSc*** package).

The stress intensity factor (SIF) computations are based on the mixed-mode J-integral formulation (Yau, Wang and Corten, *J. Appl. Mech.*, 1980; Shih and Asaro, *J. Appl. Mech.*, 1988). Let the field variables associated with a given crack problem be denoted by the superscript 1, while those corresponding to the pure mode I and pure mode II states be denoted by 21 and 22, respectively. In the above formulation, the near-tip mode I and mode II fields are used as auxiliary fields which enable the decoupling of $K_I^{(1)}$ and $K_{II}^{(1)}$ in terms of the interaction integral $M^{(1,2J)}$ ($J = 1,\,2$). The expressions for $K_I^{(1)}$ and $K_{II}^{(1)}$ are:

$$K_I^{(1)} = \frac{M^{(1,21)}}{2\alpha}\,, \quad K_{II}^{(1)} = \frac{M^{(1,22)}}{2\alpha}\,, \tag{3}$$

where

$$M^{(1,2J)} = \int_\Gamma \left\{ W^{(1,2)}\delta_{1j} - \left[\sigma_{ij}^{(1)} u_{i,1}^{(2J)} + \sigma_{ij}^{(2J)} u_{i,1}^{(1)}\right] \right\} n_j\, ds \quad (J = 1,\,2) \tag{4}$$

and

$$\alpha = \begin{cases} \frac{1-\nu^2}{E} & \text{(plane strain)} \\ \frac{1}{E} & \text{(plane stress)} \end{cases} \tag{5}$$

The domain integral representation (Moran and Shih, *Engg. Frac. Mech.*, 1987) of the above contour integral is used in the numerical computations. In its domain integral form, $M^{(1,2J)}$ becomes (see Appendix for derivation)

$$M^{(1,2I)} = -\int_A \left\{ W^{(1,2)}\delta_{1j} - \left[\sigma_{ij}^{(1)} u_{i,1}^{(2I)} + \sigma_{ij}^{(2I)} u_{i,1}^{(1)}\right] \right\} q_{,j}\, dS\,. \tag{6}$$

3 Numerical Results

3.1 Edge-Crack under Tension

An edge-cracked specimen under uniform tension ($\sigma = 1$ psi) is shown in Figure 1. The discretization, cell quadrature, and specimen geometry are indicated. The normalized SIF-results[1] for different domains (Figure 1b) is given below.

Table 1: Normalized SIF: Edge-Crack under Tension

Domains				$\frac{K_I}{\sigma\sqrt{\pi a}}$ [grid 1]	% Error	$\frac{K_I}{\sigma\sqrt{\pi a}}$ [grid 2]	% Error
a_1	a_2	b_1	b_2				
1.4	1.4	2.0	2.0	2.7946	1.1	2.7943	1.1
2.8	2.8	3.0	3.0	2.7907	1.3	2.7905	1.3
2.8	2.8	5.0	5.0	2.7987	1.0	2.7983	1.0
3.5	3.5	5.0	5.0	2.7972	1.0	2.7966	1.1
3.5	3.5	6.0	6.0	2.7982	1.0	2.7976	1.0
3.5	3.5	8.0	8.0	2.7982	1.0	2.7986	1.0

3.2 Edge Crack under Shear

An edge-crack in a finite-dimensional plate subjected to a shear stress $\tau = 1$ psi on the top surface and clamped on the bottom surface is shown in Figure 2a. The SIF-results[2] for different domains (see Figure 2b) is presented in Table 2.

Table 2: Normalized $SIFs$: Edge-Crack under Shear

Domains				K_I	% Error	K_{II}	% Error
a_1	a_2	b_1	b_2				
1.4	1.4	2.0	2.0	33.2071	2.3	4.4023	3.2
2.8	2.8	3.0	3.0	33.1767	2.4	4.4939	1.2
2.8	2.8	5.0	5.0	33.2598	2.2	4.5091	0.9
3.5	3.5	5.0	5.0	33.2791	2.1	4.5311	0.4
3.5	3.5	6.0	6.0	33.2919	2.1	4.5333	0.4
3.5	3.5	8.0	8.0	33.3036	2.0	4.5351	0.3

[1] $\frac{K_I^{\text{ref}}}{\sigma\sqrt{\pi a}} = 2.8264$ [Tada, Paris and Irwin, *The Stress Analysis of Cracks Handbook*, 1977]

[2] $K_I^{\text{ref}} = 34.0$ psi$\sqrt{in.}$, $K_{II}^{\text{ref}} = 4.55$ psi$\sqrt{in.}$, [Wilson, W.K., *Ph.D. Dissertation*, U. of Pitt., 1969]

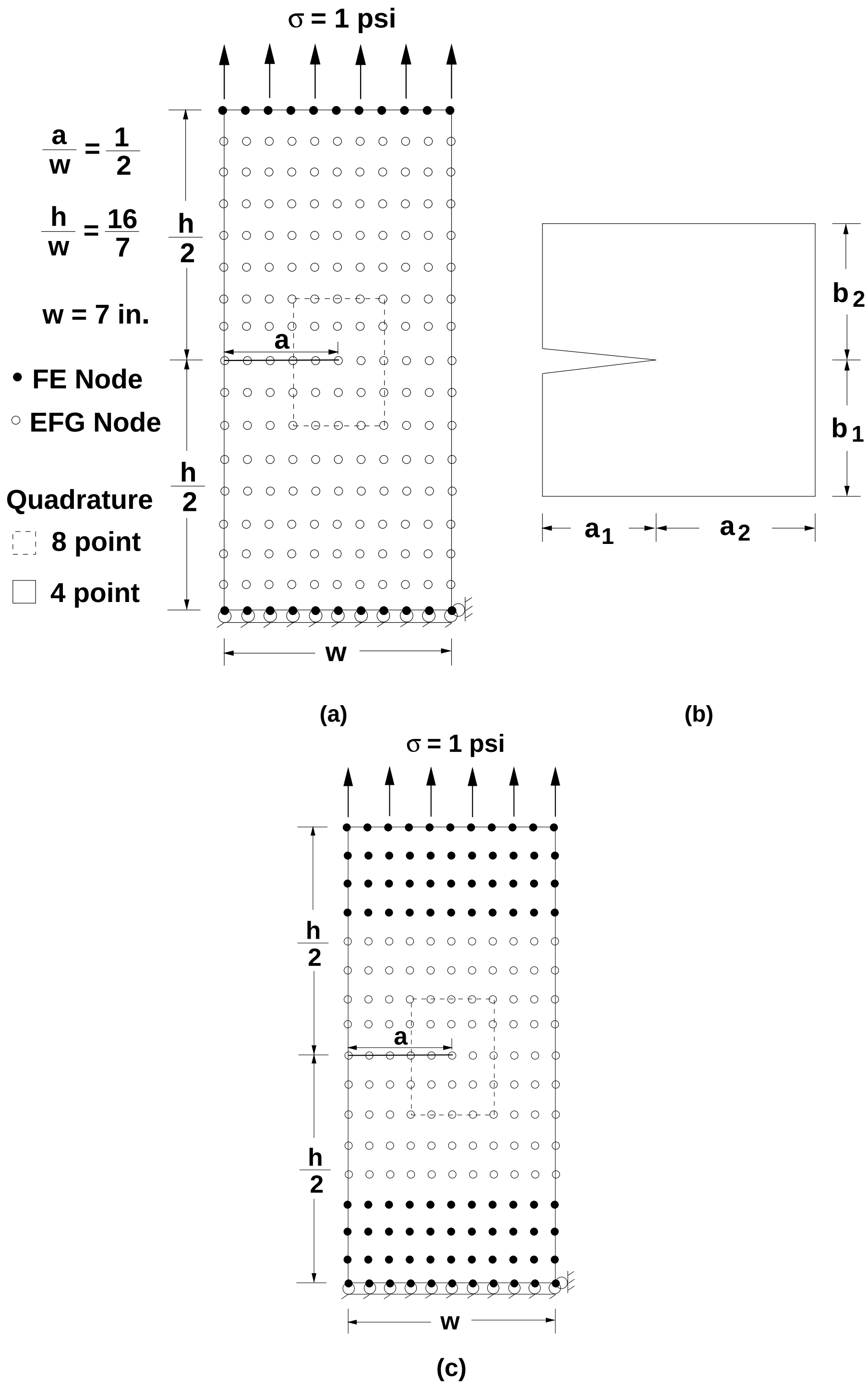


Figure 1: Edge-cracked plate under tension. (a) Discretization and specimen geometry (grid 1); (b) J-integral domain; (c) Discretization and specimen geometry (grid 2)

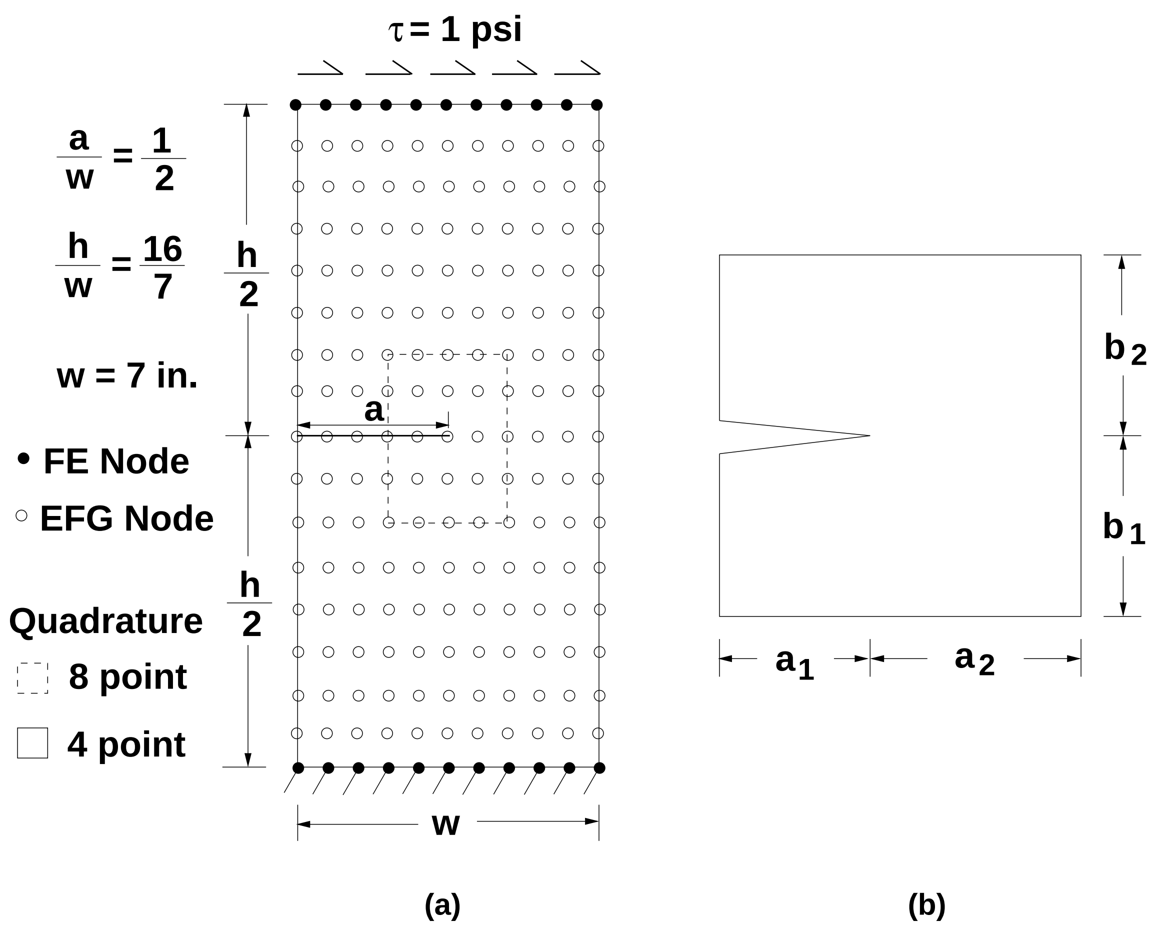
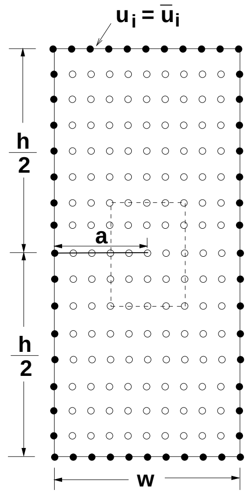


Figure 2: Edge-cracked plate under shear. (a) Discretization and specimen geometry; (b) J-integral domain

3.3 Near-Tip Stress Field



The crack-tip mode I displacement field was imposed on the boundary (FE nodes) of the edge-cracked specimen (see Figure to the left). A comparison of the numerical near-tip stress field with the exact solution is presented in Figure 3. The discontinuity in the stress field that appears in Figure 3b is due to the usage of the visibility criterion in the evaluation of the shape functions.

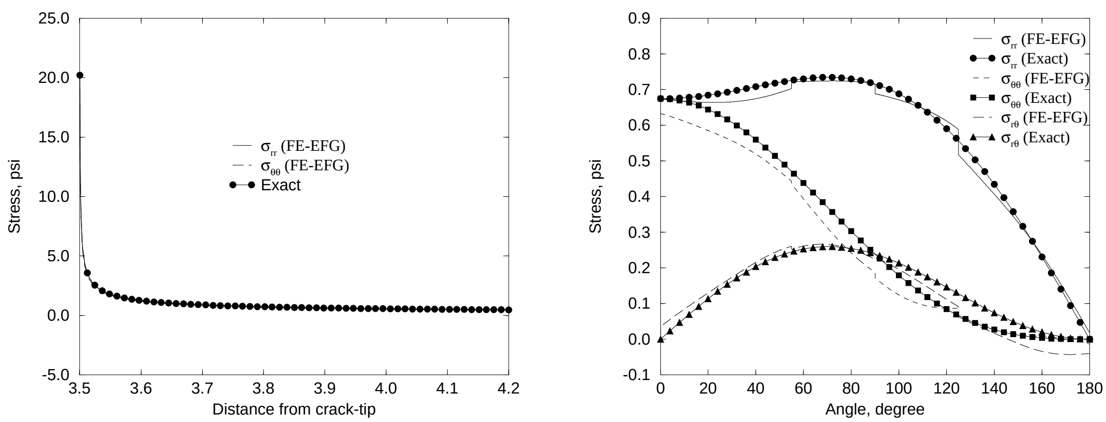


Figure 3: Comparison of the exact and numerical near-tip stress field. (a) Ahead of the crack-tip; (b) Angular variation along $r = 0.1a$ from the crack-tip.

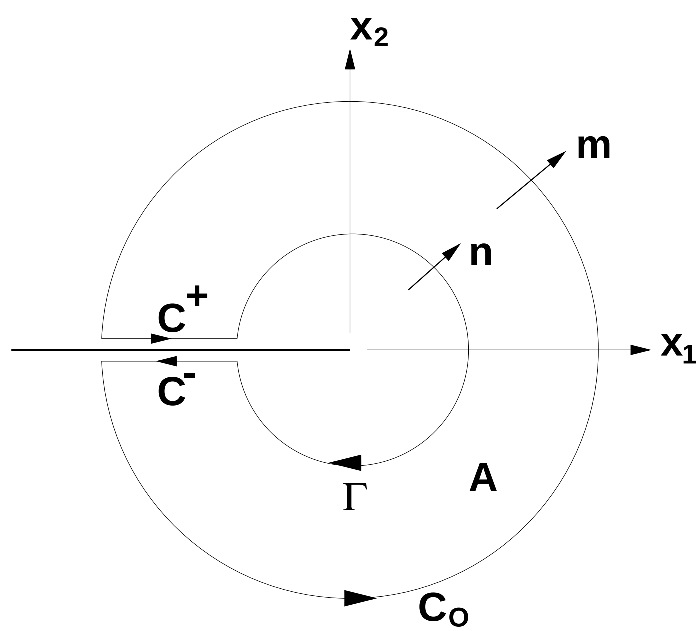
A Appendix

A.1 Derivation of the Domain Integral form for $M^{(1,2I)}$

Let

$$M^{(1,2J)} = \int_\Gamma \left\{ W^{(1,2)}\delta_{1j} - \left[\sigma_{ij}^{(1)} u_{i,1}^{(2J)} + \sigma_{ij}^{(2J)} u_{i,1}^{(1)} \right] \right\} n_j \, ds \equiv \int_\Gamma M_{1j}^J n_j \, ds \, , \tag{A.1}$$

where $J = 1,\ 2$ indicate the auxiliary fields for pure- mode I and -mode II loadings, respectively.



Now, consider a smooth test function q such that

$$q = \begin{cases} 1 & \text{on } \Gamma \\ 0 & \text{on } C_0 \\ \text{arbitrary} & \text{otherwise} \end{cases} \tag{A.2}$$

We have the following identity:

$$(M_{1j}^J q)_{,j} = M_{1j}^J q_{,j} + M_{1j,j}^J q \, . \tag{A.3}$$

Integrating the above equation over the area A and on subsequently using the divergence theorem, we obtain

$$\int_{\bar{\Gamma}} M_{1j}^J q m_j \, ds = \int_A (M_{1j}^J q_{,j} + M_{1j,j}^J q) dS \, , \tag{A.4}$$

where $\bar{\Gamma} = C^+ + C^- + \Gamma + C_0$. Since $q = 0$ on C_0, and the crack faces are traction free, the left-hand side is non-zero only on Γ. Also, noting that $m_j = -n_j$ on Γ, we have

$$-\int_\Gamma M_{1j}^J q n_j \, ds = \int_A M_{1j}^J q_{,j} \, dS + \int_A M_{1j,j}^J q dS \, . \tag{A.5}$$

We now show that M_{1j}^J is divergence-free. To this end, consider

$$\begin{aligned}
M_{1j,j}^J &= \left\{ W^{(1,2)} \delta_{1j} - \left[\sigma_{ij}^{(1)} u_{i,1}^{(2J)} + \sigma_{ij}^{(2J)} u_{i,1}^{(1)} \right] \right\}_{,j} \\
&= W_{,1}^{(1,2)} - \left[\sigma_{ij}^{(1)} u_{i,1}^{(2J)} + \sigma_{ij}^{(2J)} u_{i,1}^{(1)} \right]_{,j} \\
&= W_{,1}^{(1,2)} - \sigma_{ij,j}^{(1)} u_{i,1}^{(2J)} - \sigma_{ij}^{(1)} u_{i,1j}^{(2J)} - \sigma_{ij,j}^{(2J)} u_{i,1}^{(1)} - \sigma_{ij}^{(2J)} u_{i,1j}^{(1)} \\
&= (\sigma_{ij}^{(1)} \varepsilon_{ij}^{(2J)})_{,1} - \sigma_{ij}^{(1)} \varepsilon_{ij,1}^{(2J)} - \sigma_{ij}^{(2J)} \varepsilon_{ij,1}^{(1)} \\
&= \sigma_{ij}^{(1)} \varepsilon_{ij,1}^{(2J)} + \sigma_{ij,1}^{(1)} \varepsilon_{ij}^{(2J)} - \sigma_{ij}^{(1)} \varepsilon_{ij,1}^{(2J)} - \sigma_{ij}^{(2J)} \varepsilon_{ij,1}^{(1)} \\
&= \sigma_{ij,1}^{(1)} \varepsilon_{ij}^{(2J)} - \sigma_{ij}^{(2J)} \varepsilon_{ij,1}^{(1)} \\
&= C_{ijkl} (\varepsilon_{kl,1}^{(1)} \varepsilon_{ij}^{(2J)} - \varepsilon_{kl}^{(2J)} \varepsilon_{ij,1}^{(1)}) \\
\Rightarrow M_{1j,j}^J &= 0
\end{aligned} \tag{A.6}$$

since $C_{ijkl} = C_{klij}$ (major symmetry). In arriving at the above result, we have also used the fact that the auxiliary fields satisfy the equation of equilibrium, i.e., $\sigma_{ij,j}^{(2J)} = 0$ in the absence of body forces. On using the above result in conjunction with eqs. (**??**) and (**??**), we obtain the domain integral form:

$$\boxed{M^{(1,2J)} = -\int_A M_{1j}^J q_{,j} \, dS \, .} \tag{A.7}$$