PUFEM in One-Dimension

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1 Problem Statement

Consider the following boundary-value problem:

$$\begin{aligned} -\mathcal{L}u + \lambda^2 u &= q \quad \text{in } \Omega\,, \\ u(0) &= 0\,, \\ u(L) &= 0\,, \end{aligned} \tag{1}$$

where $\mathcal{L} \equiv d^2/dx^2$ is the Laplacian operator in 1D, and $\Omega = \{x \mid x \in (0, L)\}$.

2 Discrete System

Let the trial function $u^h(x)$ be given by:

$$u^h(x) = \sum_I \hat{\boldsymbol{\phi}}_I \mathbf{a}^I\,, \tag{2}$$

where $\hat{\boldsymbol{\phi}}_I = \phi_I \{1 \;\; x \;\; \ldots\}$ is the shape function vector at node I, and $\mathbf{a}^I$ are the corresponding vector of unknown coefficients at node I. We choose $\phi_I = w_I / \sum_K w_K$ to be the partition of unity for patch Ω_I. By substituting the trial and test functions in the weak form, the following discrete system of equations is obtained:

$$\mathbf{K}\,\mathbf{a} = \mathbf{f} \tag{3}$$

where

$$\begin{aligned} \mathbf{K}_{IJ} &= \int_\Omega (\mathbf{B}_I^T \mathbf{B}_J + \lambda^2 \hat{\boldsymbol{\phi}}_I^T \hat{\boldsymbol{\phi}}_J)\, d\Omega \\ \mathbf{f}_I &= \int_\Omega \hat{\boldsymbol{\phi}}_I^T q\, d\Omega \end{aligned} \tag{4}$$

3 Discretization and Numerical Results

The domain is discretized by n nodes (Figure 1). Let the spacing between adjacent nodes $h = \frac{L}{n-1}$. The nodal coordinates are $x_j = (j-1)h,\ j = 1,\ 2,\ \ldots,\ n$. Also define $x_0 = -h$ and $x_{n+1} = 1 + h$. Consider patches $\Omega_j = \{x \mid x \in (x_{j-1}, x_{j+1})\}$. Let the local space on each patch be denoted by V_j.

The partition of unity $\phi_I(x)$ was represented by considering a quartic spline weight function $w(x)$. Five-point Gauss quadrature was used in the numerical integration. In Examples 1, 2, and 3, the domain was discretized by 11 nodes ($n = 11$), while 101 nodes were used in Example 4. A conjugate gradient linear equation solver with block Jacobi preconditioner (***PETSc*** package) was used in the computations.

3.1 Example 1

Consider the BVP (1) for

$$\lambda = 0\,,\ q(x) = 2\,,\ \text{and}\ L = 1\,. \tag{5}$$

The exact solution is:

$$\begin{aligned} u(x) &= x - x^2\,, \\ \frac{du(x)}{dx} &= 1 - 2x\,. \end{aligned} \tag{6}$$

The local spaces V_i are chosen as:

$$\begin{aligned} V_1 &= \text{span}\ \{x,\ -x^2\} \quad \text{on}\ \Omega_1 \cap \Omega \\ V_j &= \text{span}\ \{1,\ x - x_j,\ (x - x_j)^2\} \quad \text{on}\ \Omega_\mathrm{j} \cap \Omega,\ \mathrm{j} = 2,\ 3,\ \ldots, \mathrm{n} - 1 \\ V_{11} &= \text{span}\ \{x,\ -x^2\} \quad \text{on}\ \Omega_{11} \cap \Omega \end{aligned} \tag{7}$$

The displacement and strain results are shown in Figures 2 and 3, respectively. Convergence was attained in one iteration: Residual norm $||\mathbf{Ka} - \mathbf{f}||_{\mathcal{L}_2}$ = 2.1E-15.

3.2 Example 2

Consider the BVP (1) for

$$\lambda = 0\,,\ q(x) = \frac{3}{4\sqrt{x}}\,,\ \text{and}\ L = 1\,. \tag{8}$$

The exact solution is:

$$\begin{aligned} u(x) &= x - x^{3/2}\,, \\ \frac{du(x)}{dx} &= 1 - \frac{3}{2}\sqrt{x}\,. \end{aligned} \tag{9}$$

The local spaces V_i are chosen as:

$$\begin{aligned}
V_1 &= \text{span}\ \{x,\ -x^{3/2}\} \quad \text{on } \Omega_1 \cap \Omega \\
V_j &= \text{span}\ \{1,\ x - x_j,\ x^{3/2}\} \quad \text{on } \Omega_\mathrm{j} \cap \Omega,\ \mathrm{j} = 2,\ 3,\ \ldots, \mathrm{n} - 1 \\
V_{11} &= \text{span}\ \{x,\ -x^{3/2}\} \quad \text{on } \Omega_{11} \cap \Omega
\end{aligned} \tag{10}$$

The displacement and strain results are shown in Figures 4 and 5, respectively. Convergence was attained in one iteration: Residual norm $||\mathbf{Ka} - \mathbf{f}||_{\mathcal{L}_2} = 9.2\text{E-}15$.

3.3 Example 3

Consider the BVP (1) for

$$\lambda = 1\,,\ q(x) = 2\,,\ \text{and } L = 1\,. \tag{11}$$

The exact solution for $q = 2$ is:

$$\begin{aligned}
u(x) &= 2(1 - \cosh \lambda x) - \frac{2(1 - \cosh \lambda L)}{\sinh \lambda L} \sinh \lambda x\,, \\
\frac{du(x)}{dx} &= -2\lambda \sinh \lambda x - \frac{2\lambda(1 - \cosh \lambda L)}{\sinh \lambda L} \cosh \lambda x\,.
\end{aligned} \tag{12}$$

The local spaces V_i are chosen as:

$$\begin{aligned}
V_1 &= \text{span}\ \{\sinh x, 1 - \cosh x\} \quad \text{on } \Omega_0 \cap \Omega \\
V_j &= \text{span}\ \{1, \sinh(x - x_j), \cosh(x - x_j)\} \quad \text{on } \Omega_\mathrm{j} \cap \Omega,\ \mathrm{j} = 2,\ 3,\ \ldots, \mathrm{n} - 1 \\
V_{11} &= \text{span}\ \{2(1 - \cosh x),\ -\frac{2(1 - \cosh L)}{\sinh L} \sinh x\} \quad \text{on } \Omega_{11} \cap \Omega
\end{aligned} \tag{13}$$

The displacement and strain results are presented in Figures 6 and 7, respectively. Convergence was attained in one iteration: Residual norm $||\mathbf{Ka} - \mathbf{f}||_{\mathcal{L}_2} = 1.4\text{E-}13$.

3.4 Example 4

Consider the BVP (1) for

$$\lambda = 1\,,\ q(x) = 2\,,\ \text{and } L = 10\,. \tag{14}$$

The exact solution is given in eq. (**??**). The local spaces V_i are chosen as:

$$\begin{aligned}
V_1 &= \text{span}\ \{2(1 - \cosh x),\ -\frac{2(1 - \cosh L)}{\sinh L} \sinh x\} \quad \text{on } \Omega_1 \cap \Omega \\
V_j &= \text{span}\ \{1, \sinh(x - x_j), \cosh(x - x_j)\} \quad \text{on } \Omega_\mathrm{j} \cap \Omega,\ \mathrm{j} = 2,\ 3,\ \ldots, \mathrm{n} - 1 \\
V_{101} &= \text{span}\ \{2(1 - \cosh x),\ -\frac{2(1 - \cosh L)}{\sinh L} \sinh x\} \quad \text{on } \Omega_{101} \cap \Omega
\end{aligned} \tag{15}$$

The displacement and strain results are shown in Figures 8 and 9, respectively. Convergence was attained in one iteration: Residual norm $||\mathbf{Ka} - \mathbf{f}||_{\mathcal{L}_2} = 1.1\text{E-}5$.

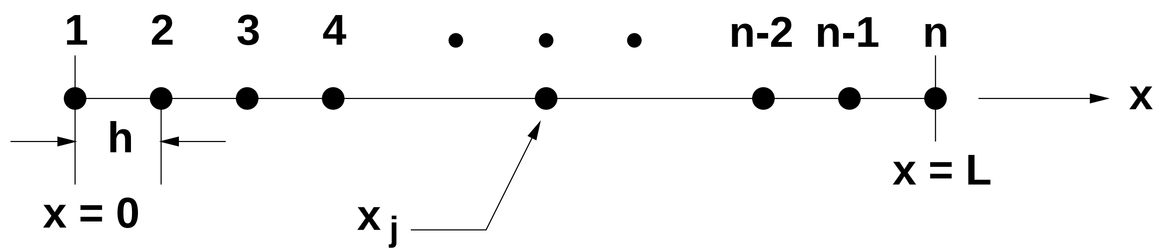

Figure 1: Nodal Discretization

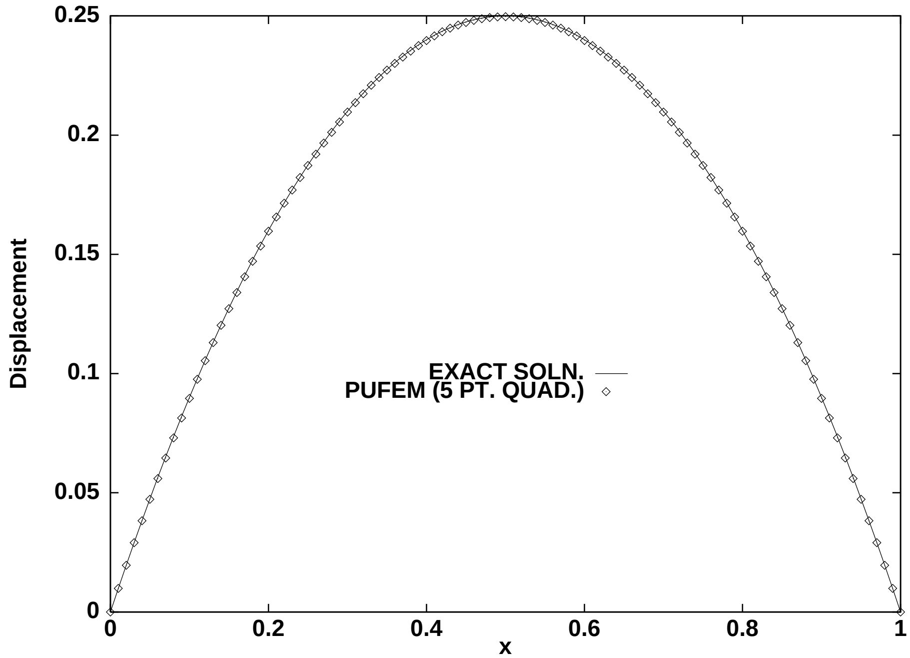

Figure 2: Displacement along the 1D bar for example 1

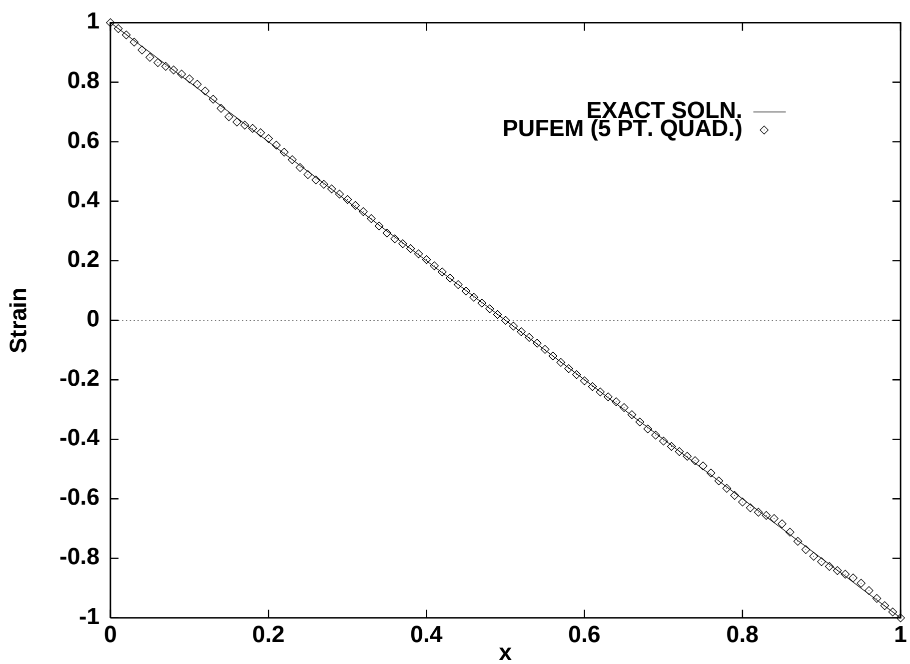

Figure 3: Strain along the 1D bar for example 1

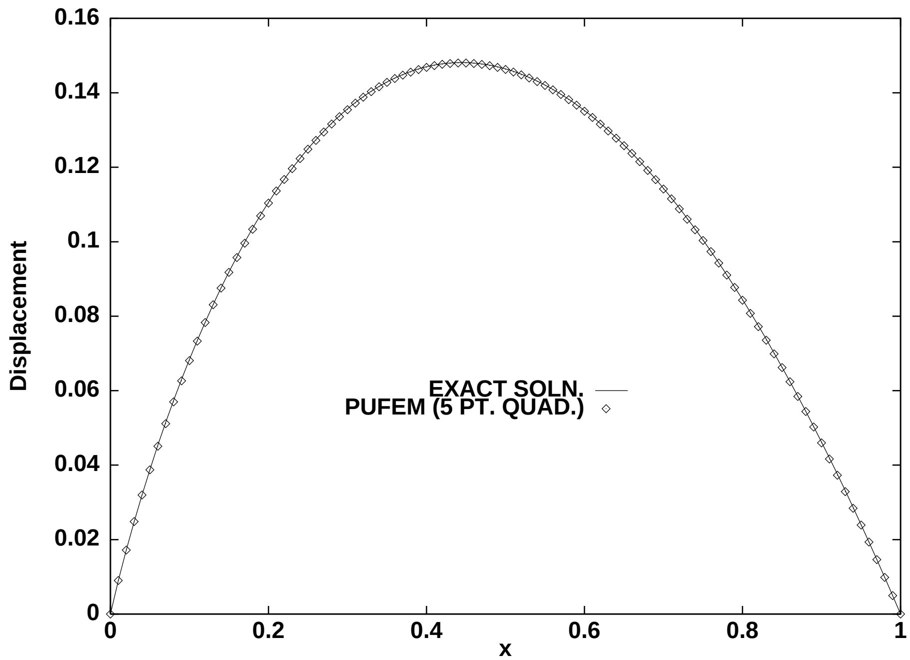

Figure 4: Displacement along the 1D bar for example 2

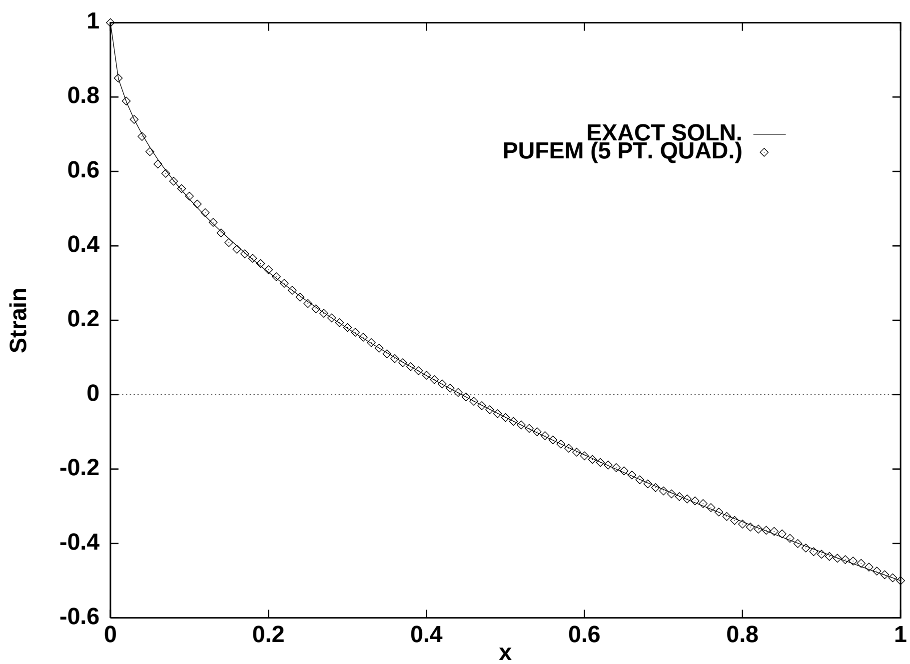

Figure 5: Strain along the 1D bar for example 2

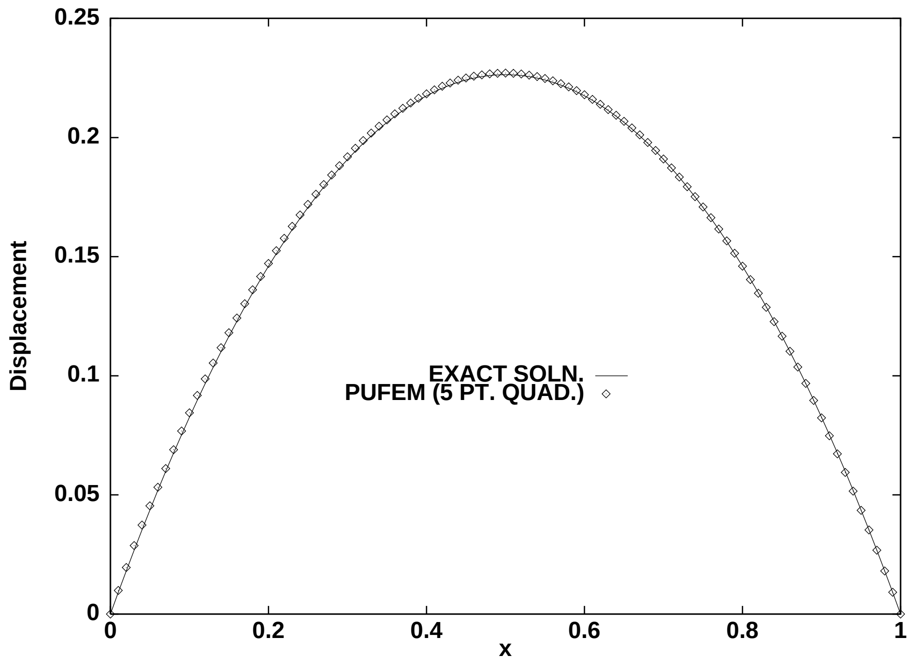

Figure 6: Displacement along the 1D bar for example 3

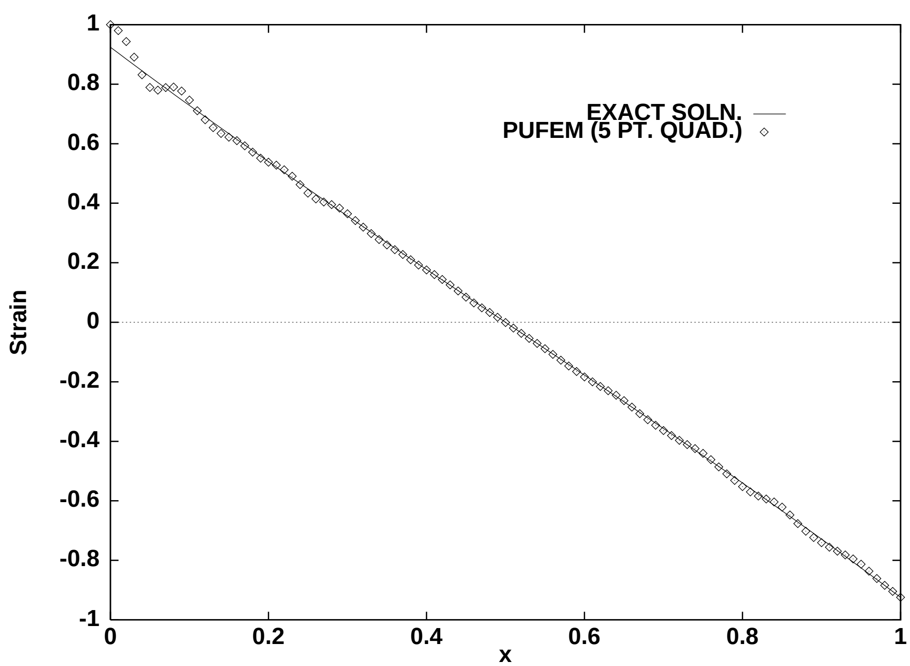

Figure 7: Strain along the 1D bar for example 3

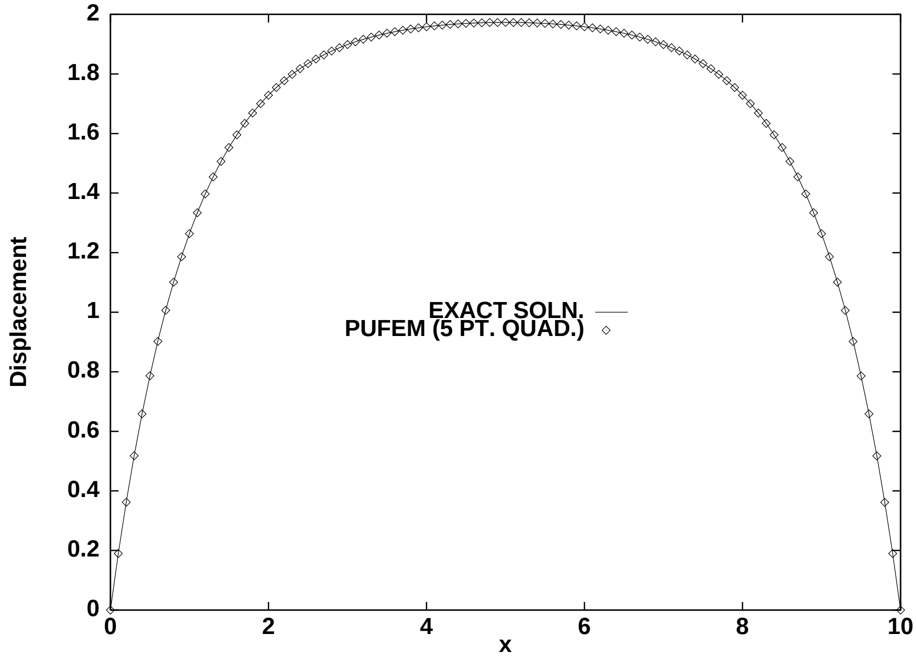

Figure 8: Displacement along the 1D bar for example 4

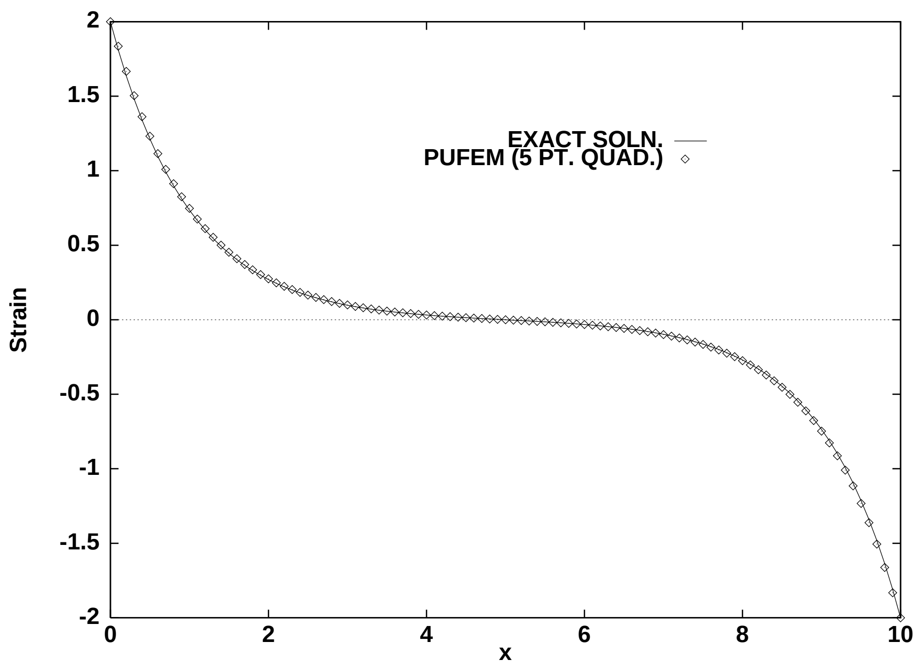

Figure 9: Strain along the 1D bar for example 4