PUFEM in One-Dimension

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1 Problem Statement

Consider the following boundary-value problem:

\[-\mathcal{L}u + \lambda^2 u = q \text{ in } \Omega,\]
\[
u(0) = 0,\]
\[
u(L) = 0,\]  \hspace{1cm} (1)

where \(\mathcal{L} \equiv d^2/dx^2\) is the Laplacian operator in 1D, and \(\Omega = \{x | x \in (0, L)\}\).

2 Discrete System

Let the trial function \(u^h(x)\) be given by:

\[u^h(x) = \sum_I \hat{\phi}_I a^I,\]  \hspace{1cm} (2)

where \(\hat{\phi}_I = \phi_I \{1 \ x \ \ldots\}\) is the shape function vector at node \(I\), and \(a^I\) are the corresponding vector of unknown coefficients at node \(I\). We choose \(\phi_I = w_I/\sum_K w_K\) to be the partition of unity for patch \(\Omega_I\). By substituting the trial and test functions in the weak form, the following discrete system of equations is obtained:

\[K a = f\]  \hspace{1cm} (3)

where

\[K_{IJ} = \int_\Omega (B_I^T B_J + \lambda^2 \hat{\phi}_I^T \hat{\phi}_J) \, d\Omega\]
\[f_I = \int_\Omega \hat{\phi}_I^T q \, d\Omega\]  \hspace{1cm} (4)
3 Discretization and Numerical Results

The domain is discretized by $n$ nodes (Figure 1). Let the spacing between adjacent nodes $h = \frac{L}{n-1}$. The nodal coordinates are $x_j = (j - 1)h$, $j = 1, 2, \ldots, n$. Also define $x_0 = -h$ and $x_{n+1} = 1 + h$. Consider patches $\Omega_j = \{ x \mid x \in (x_{j-1}, x_{j+1}) \}$. Let the local space on each patch be denoted by $V_j$.

The partition of unity $\phi_I(x)$ was represented by considering a quartic spline weight function $w(x)$. Five-point Gauss quadrature was used in the numerical integration. In Examples 1, 2, and 3, the domain was discretized by 11 nodes ($n = 11$), while 101 nodes were used in Example 4. A conjugate gradient linear equation solver with block Jacobi preconditioner ($\texttt{PETSc}$ package) was used in the computations.

3.1 Example 1

Consider the BVP (1) for

$$\lambda = 0, \quad q(x) = 2, \quad \text{and} \quad L = 1.$$  \hfill (5)

The exact solution is:

$$u(x) = x - x^2,$$

$$\frac{du(x)}{dx} = 1 - 2x.$$  \hfill (6)

The local spaces $V_i$ are chosen as:

$$V_1 = \text{span} \{ x, -x^2 \} \quad \text{on} \quad \Omega_1 \cap \Omega \,$$

$$V_j = \text{span} \{ 1, x - x_j, (x - x_j)^2 \} \quad \text{on} \quad \Omega_j \cap \Omega, \quad j = 2, 3, \ldots, n - 1$$ \hfill (7)

$$V_{11} = \text{span} \{ x, -x^2 \} \quad \text{on} \quad \Omega_{11} \cap \Omega$$

The displacement and strain results are shown in Figures 2 and 3, respectively. Convergence was attained in one iteration: Residual norm $||Ku - f||_{L_2} = 2.1E-15$.

3.2 Example 2

Consider the BVP (1) for

$$\lambda = 0, \quad q(x) = \frac{3}{4\sqrt{x}}, \quad \text{and} \quad L = 1.$$  \hfill (8)

The exact solution is:

$$u(x) = x - x^{3/2},$$

$$\frac{du(x)}{dx} = 1 - \frac{3}{2\sqrt{x}}.$$  \hfill (9)
The local spaces $V_i$ are chosen as:

\[
V_1 = \text{span} \{ x, -x^{3/2} \} \quad \text{on} \quad \Omega_1 \cap \Omega \\
V_j = \text{span} \{ 1, x - x_j, x^{3/2} \} \quad \text{on} \quad \Omega_j \cap \Omega, \ j = 2, 3, \ldots, n - 1 \\
V_{11} = \text{span} \{ x, -x^{3/2} \} \quad \text{on} \quad \Omega_{11} \cap \Omega
\]

The displacement and strain results are shown in Figures 4 and 5, respectively. Convergence was attained in one iteration: Residual norm $||\mathbf{K} \mathbf{a} - \mathbf{f}||_{\mathcal{L}_2} = 9.2 \times 10^{-15}$.

### 3.3 Example 3

Consider the BVP (1) for

\[
\lambda = 1, \ q(x) = 2, \ \text{and} \ L = 1.
\]

The exact solution for $q = 2$ is:

\[
u(x) = 2(1 - \cosh \lambda x) - \frac{2(1 - \cosh \lambda L)}{\sinh \lambda L} \sinh \lambda x, \\
\frac{d\nu(x)}{dx} = -2\lambda \sinh \lambda x - \frac{2\lambda(1 - \cosh \lambda L)}{\sinh \lambda L} \cosh \lambda x.
\]

The local spaces $V_i$ are chosen as:

\[
V_1 = \text{span} \{ \sinh x, 1 - \cosh x \} \quad \text{on} \quad \Omega_0 \cap \Omega \\
V_j = \text{span} \{ 1, \sinh(x - x_j), \cosh(x - x_j) \} \quad \text{on} \quad \Omega_j \cap \Omega, \ j = 2, 3, \ldots, n - 1 \\
V_{11} = \text{span} \{ 2(1 - \cosh x), -\frac{2(1 - \cosh L)}{\sinh L} \sinh x \} \quad \text{on} \quad \Omega_{11} \cap \Omega
\]

The displacement and strain results are presented in Figures 6 and 7, respectively. Convergence was attained in one iteration: Residual norm $||\mathbf{K} \mathbf{a} - \mathbf{f}||_{\mathcal{L}_2} = 1.4 \times 10^{-13}$.

### 3.4 Example 4

Consider the BVP (1) for

\[
\lambda = 1, \ q(x) = 2, \ \text{and} \ L = 10.
\]

The exact solution is given in eq. ( ??). The local spaces $V_i$ are chosen as:

\[
V_1 = \text{span} \{ 2(1 - \cosh x), -\frac{2(1 - \cosh L)}{\sinh L} \sinh x \} \quad \text{on} \quad \Omega_1 \cap \Omega \\
V_j = \text{span} \{ 1, \sinh(x - x_j), \cosh(x - x_j) \} \quad \text{on} \quad \Omega_j \cap \Omega, \ j = 2, 3, \ldots, n - 1 \\
V_{101} = \text{span} \{ 2(1 - \cosh x), -\frac{2(1 - \cosh L)}{\sinh L} \sinh x \} \quad \text{on} \quad \Omega_{101} \cap \Omega
\]
The displacement and strain results are shown in Figures 8 and 9, respectively. Convergence was attained in one iteration: Residual norm $\|Ku - f\|_2 = 1.1E-5$.

Figure 1: Nodal Discretization
Figure 2: Displacement along the 1D bar for example 1

Figure 3: Strain along the 1D bar for example 1
Figure 4: Displacement along the 1D bar for example 2

Figure 5: Strain along the 1D bar for example 2
Figure 6: Displacement along the 1D bar for example 3

Figure 7: Strain along the 1D bar for example 3
Figure 8: Displacement along the 1D bar for example 4

Figure 9: Strain along the 1D bar for example 4