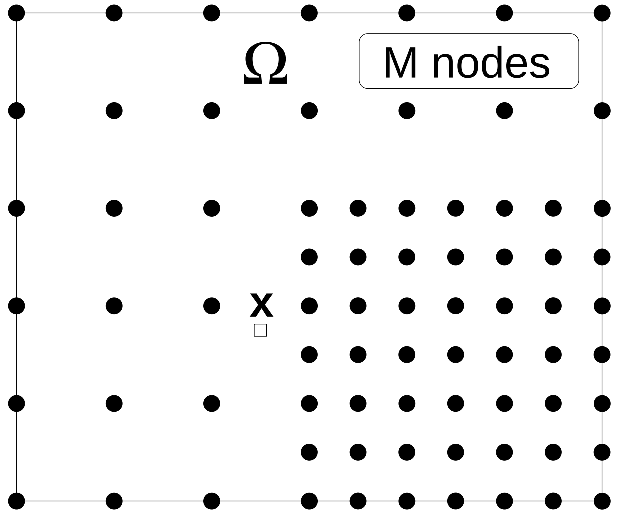
Galerkin Methods

Finite Element Methods

Meshless Methods [DEM, EFG, RKPM, PUM, Hp-Clouds, . .]

Natural Element Method

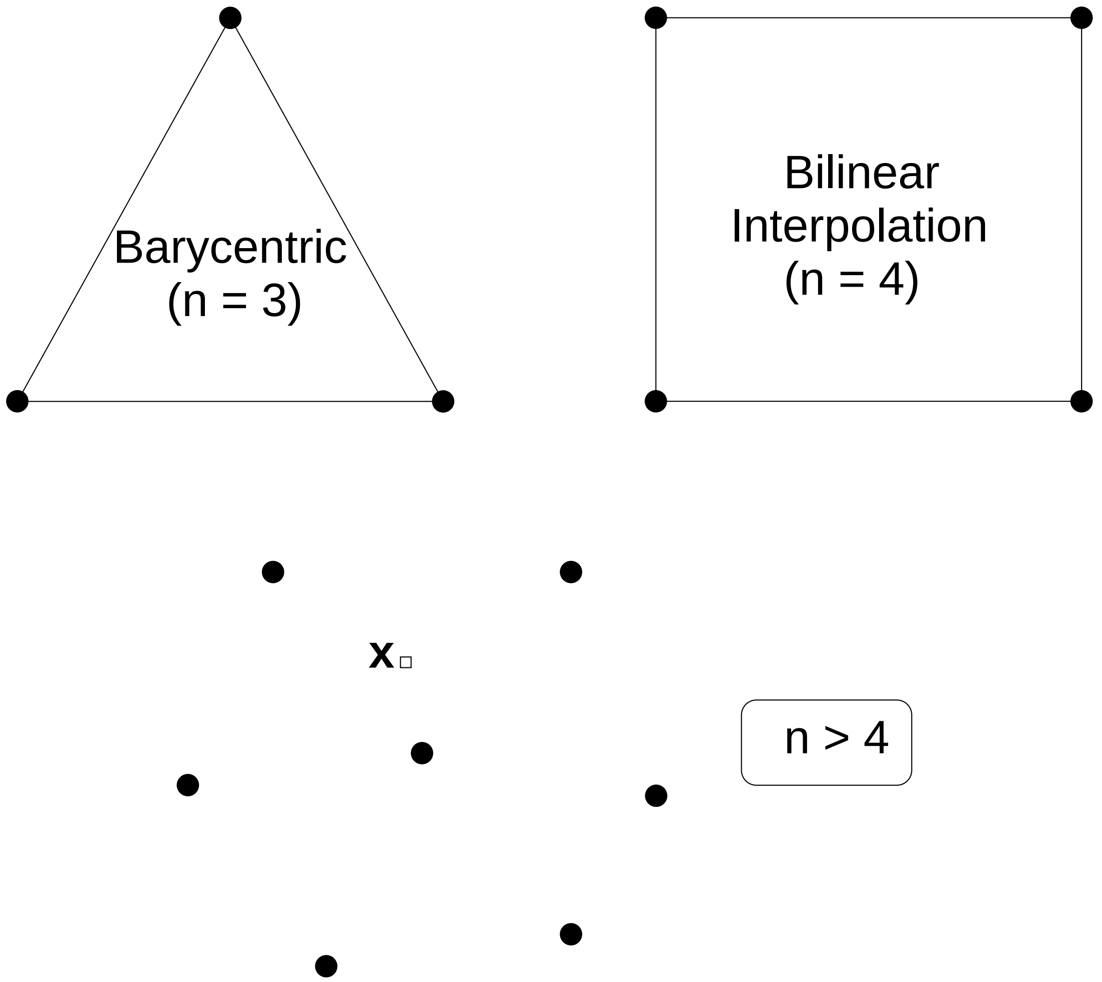
(Braun and Sambridge, 1995)



$$u^h(\mathbf{x}) = \sum_{I=1}^{n} \phi_I(\mathbf{x}) u_I, \ n << M.$$

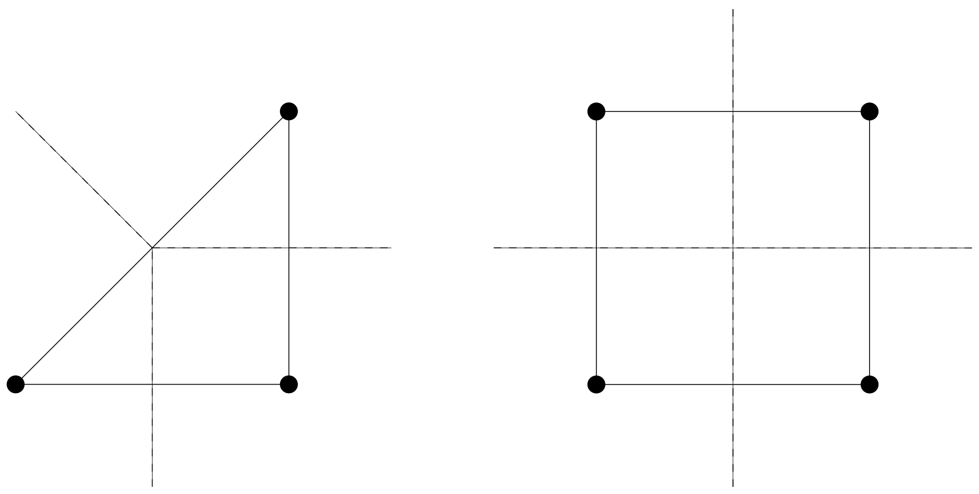
Natural Neighbor Interpolation

- Sibson (1980) introduced natural neighbor coordinates as a means for data interpolation and smoothing.



Voronoi Diagrams

- Engineering, physical and applied sciences, etc.
- Computer science and computational geometry
- Mathematics



Properties

- Positivity and Interpolation

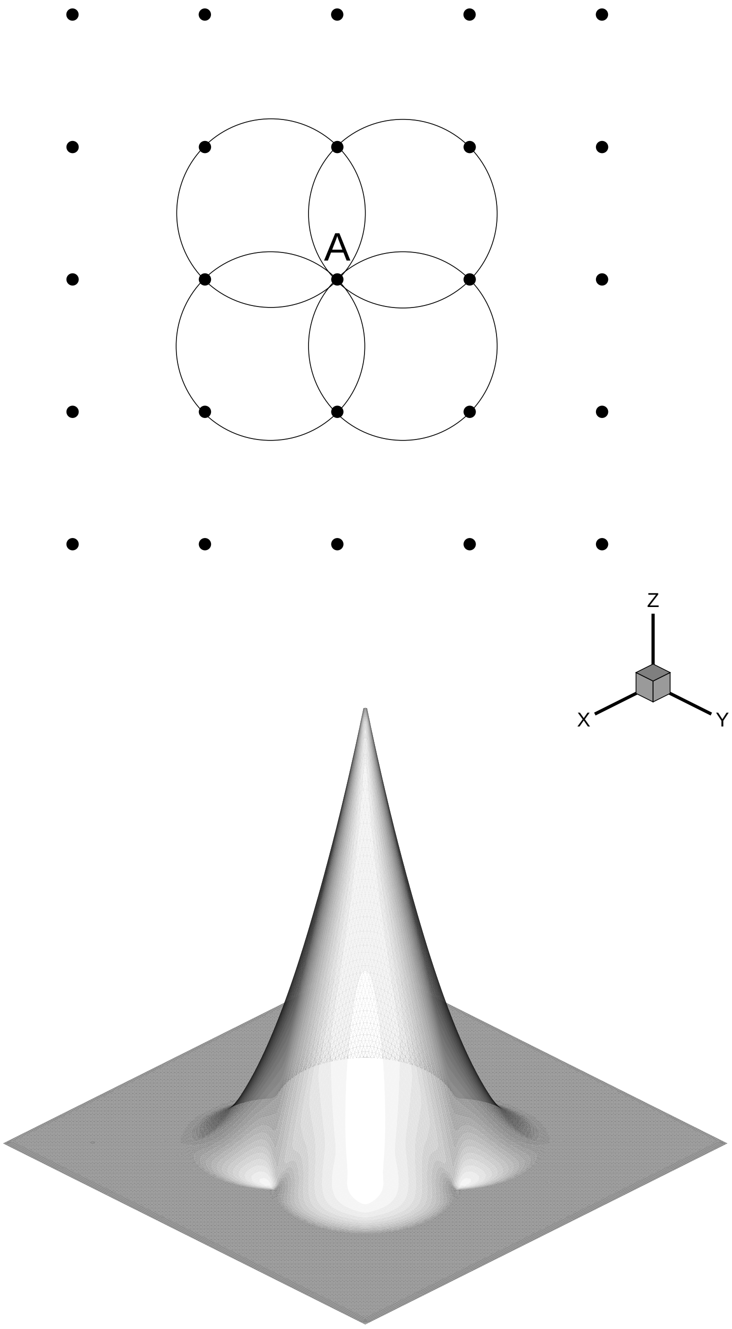
$$\boxed{0 \leq \phi_I(\mathbf{x}) \leq 1, \quad \phi_I(\mathbf{x}_J) = \delta_{IJ}, \quad \mathbf{u}^h(\mathbf{x}_I) = \mathbf{u}_I}$$

- Partition of Unity and Linear Completeness

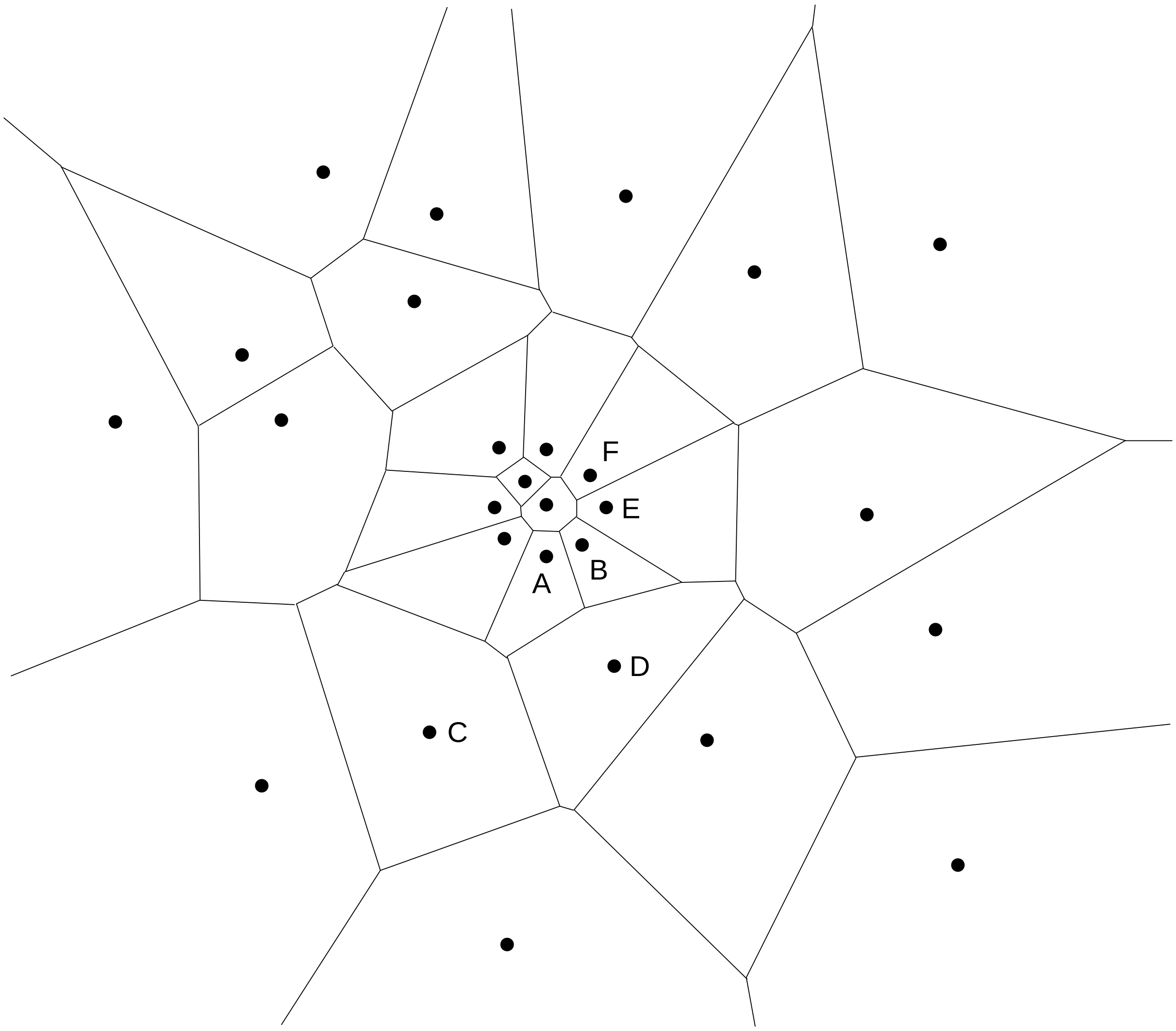
$$\boxed{\sum_{I=1}^{n} \phi_I(\mathbf{x}) = 1, \quad \mathbf{x} = \sum_{I=1}^{n} \phi_I(\mathbf{x})\mathbf{x}_I}$$

- Linear approximation along the boundary of a convex domain Ω, and hence in NEM, essential boundary conditions can be directly imposed on the nodes.

- In 1D, natural neighbor interpolation is identical to linear finite elements.



Regular Nodal Grid and Shape function $\phi_A(\mathbf{x})$ for Node A



Nodal set N and Voronoi Diagram $V(N)$

Weak Form (Discrete)

$$\boxed{\begin{gathered}\text{Find } \mathbf{u}^h \in \mathbf{V}^h \subset \mathbf{V} \ \text{ such that } \forall\, \mathbf{v}^h \in \mathbf{V}_0^h \subset \mathbf{V}_0 \\ \int_{\Omega^h} \boldsymbol{\sigma}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\mathbf{v}^h) = \int_{\Omega^h} \mathbf{b} \cdot \mathbf{v}\, d\Omega + \int_{\Gamma_t^h} \bar{\mathbf{t}} \cdot \mathbf{v}\, d\Gamma \\ \mathbf{V} = \left(H^1(\Omega)\right)^2,\ \mathbf{V}_0 = \left(H_0^1(\Omega)\right)^2\end{gathered}}$$

NEM Trial and Test Functions

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I=1}^{n} \phi_I(\mathbf{x}) \mathbf{u}_I, \quad \mathbf{v}^h(\mathbf{x}) = \sum_{I=1}^{n} \phi_I(\mathbf{x}) \mathbf{v}_I$$

Discrete System

$$\boldsymbol{K}\mathbf{d} = \mathbf{f},$$

$$\mathbf{K}_{IJ} = \int_{\Omega^h} \mathbf{B}_I^T \mathbf{C} \mathbf{B}_J \, d\Omega,$$

$$\mathbf{f}_I = \int_{\Gamma_t^h} \phi_I \bar{\mathbf{t}}\, d\Gamma + \int_{\Omega^h} \phi_I \mathbf{b}\, d\Omega + \int_{\Omega^h} \boldsymbol{B}_I^T \mathbf{C} \boldsymbol{\varepsilon}^*\, d\Omega$$

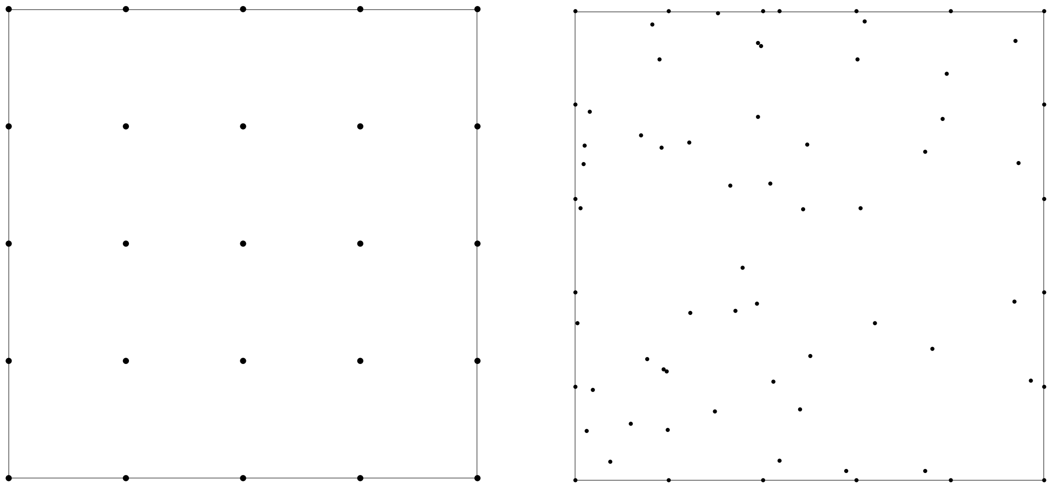
NEM Approximation Spaces

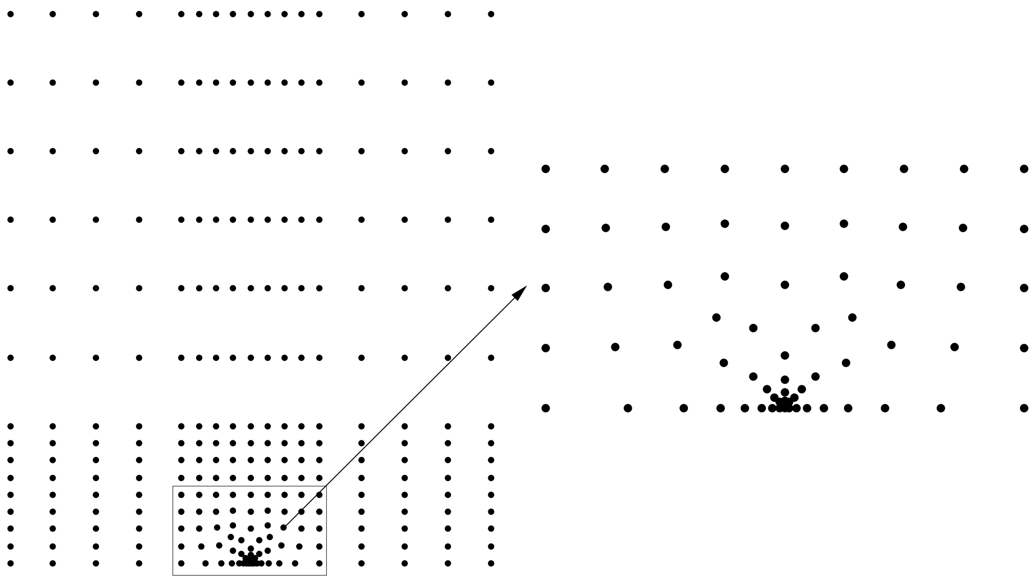
- Condition number κ^h of the eigenvalue problem

$$\boxed{\mathbf{M}\hat{\mathbf{d}} = \lambda^h \hat{\mathbf{d}}, \quad \mathbf{M}_{IJ} = \int_{\Omega^h} \phi_I \phi_J \, d\Omega}$$

Grids	Nodes	κ^h_{NEM} Quadrature 3	κ^h_{NEM} Quadrature 25	κ^h_{FEM}
Uniform	25	3.99	3.83	7.00
	289	3.99	3.83	7.00
	4225	3.99	3.83	7.00
Non-uniform	70	50.86	35.65	43.17
	278	2148.97	2146.45	3422.31

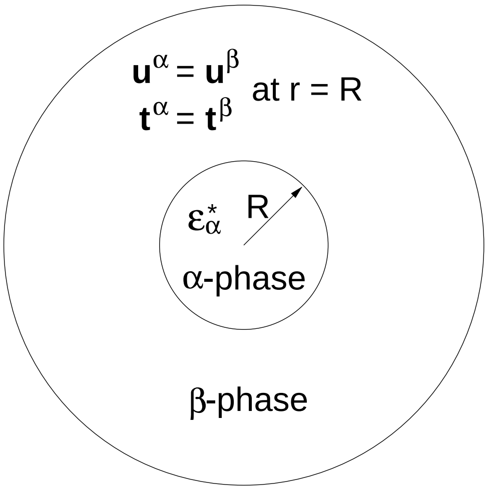
Nodal Grids for Eigenanalysis





Numerical Results

I. Infinite Plate with an Inclusion



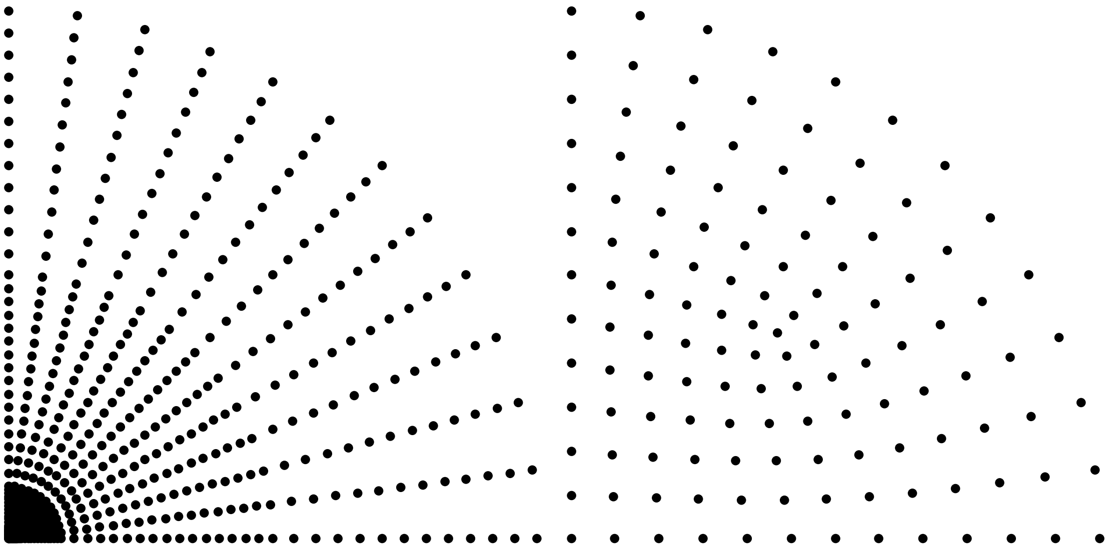
Model

Material Constants

$$\lambda^\alpha = 497.16, \ \mu^\alpha = 390.63,$$

$$\lambda^\beta = 656.79, \ \mu^\beta = 338.35,$$

$$\boldsymbol{\varepsilon}^*_\alpha = 0.01(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2)$$

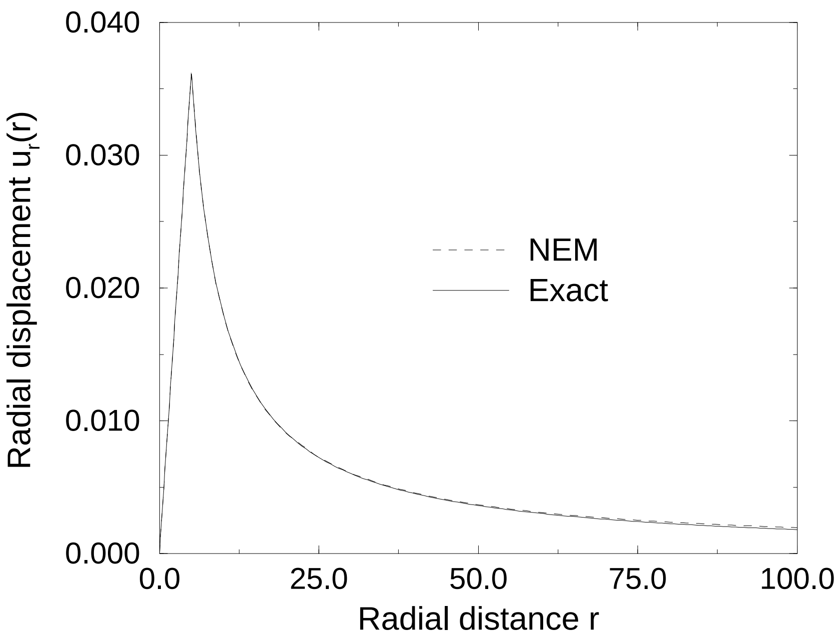


Nodal Discretization (Quarter model)

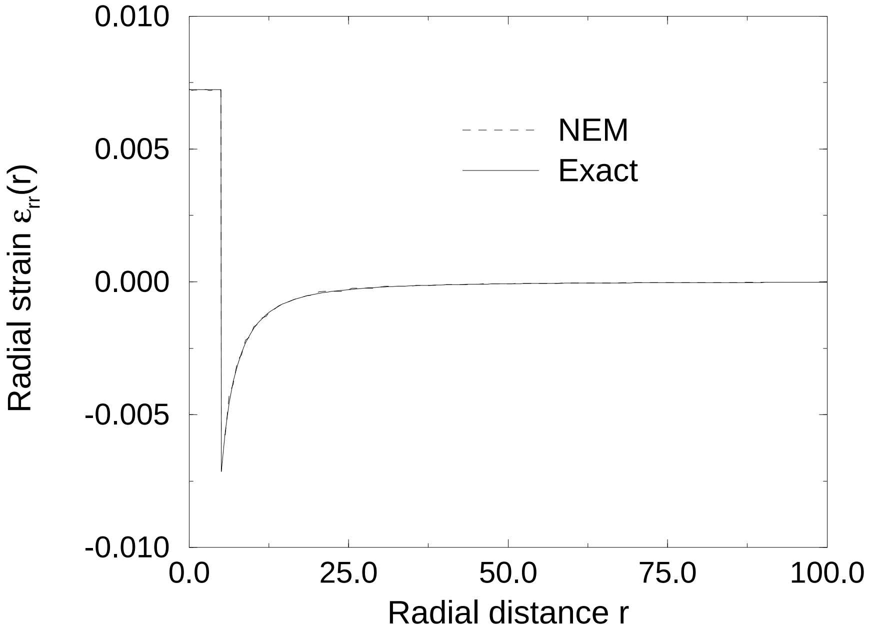
Exact Solution

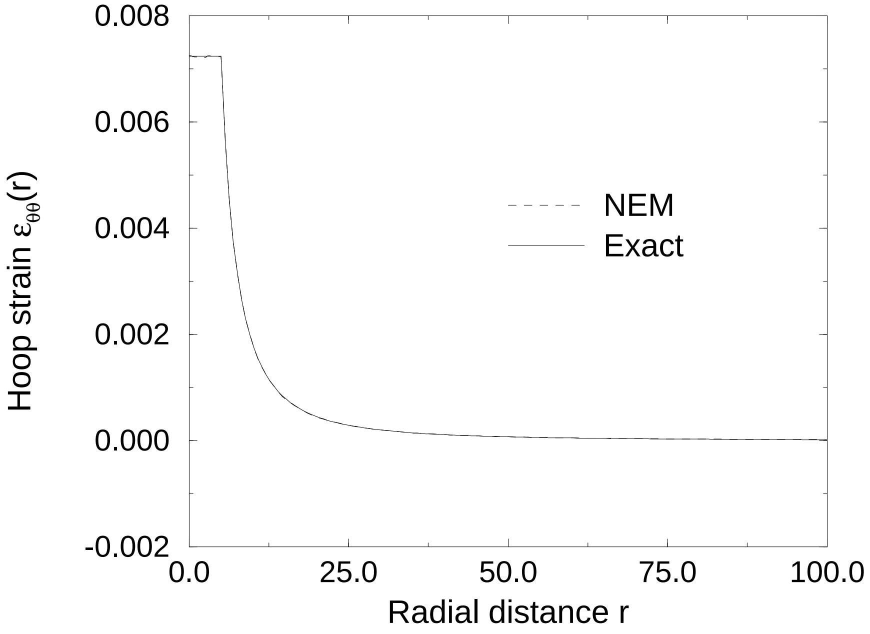
$$u_r(r) = \begin{cases} C_1 r & r \le R, \\ C_1 \dfrac{R^2}{r} & r \ge R, \end{cases}$$

$$C_1 = \frac{(\mu^\alpha + \lambda^\alpha)\bar{\varepsilon}^*_\alpha}{\mu^\alpha + \lambda^\alpha + \mu^\beta}$$

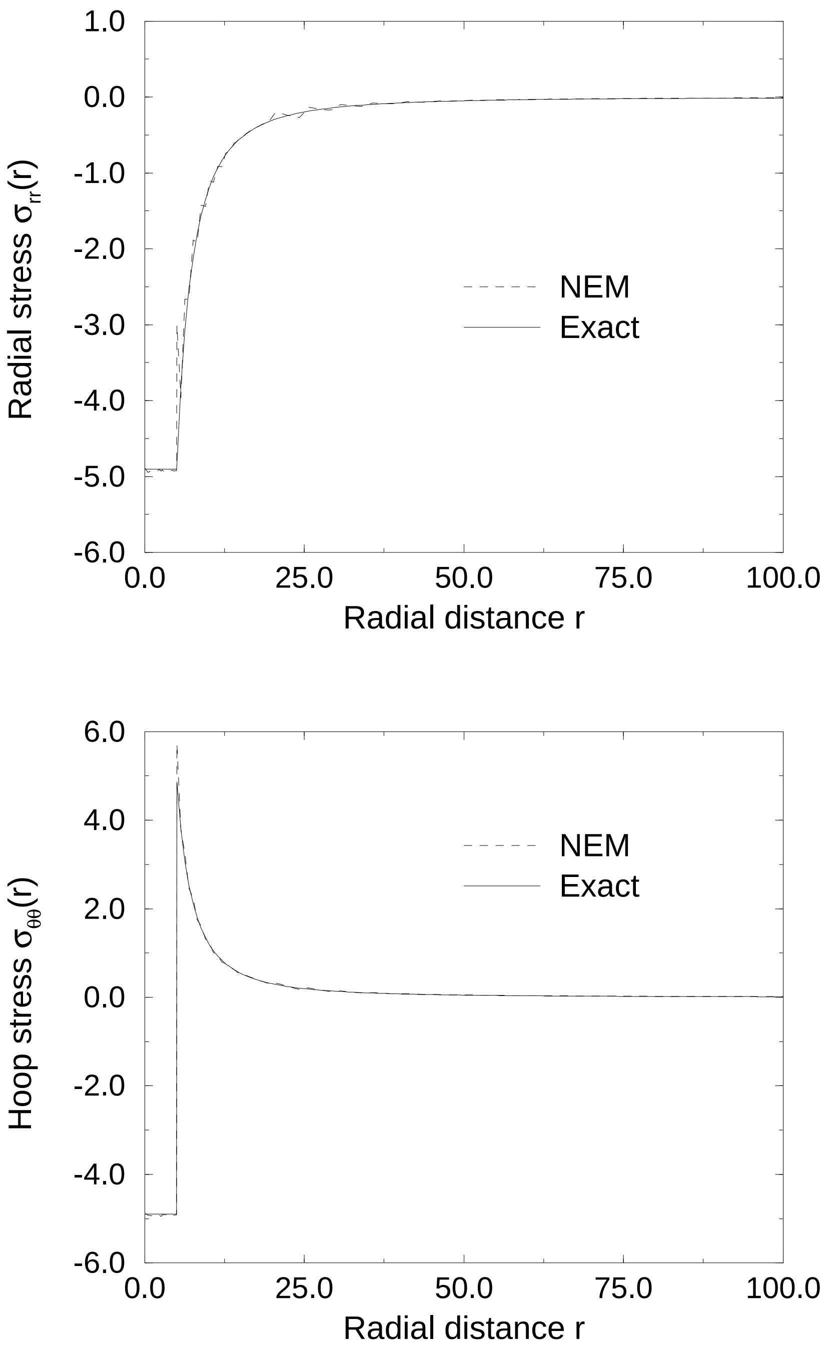


Variation of u_r Along a Radial Line



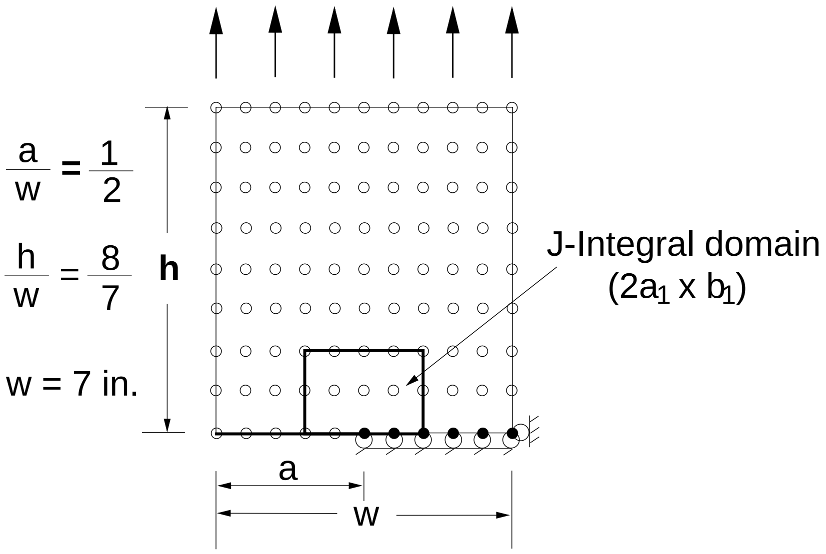


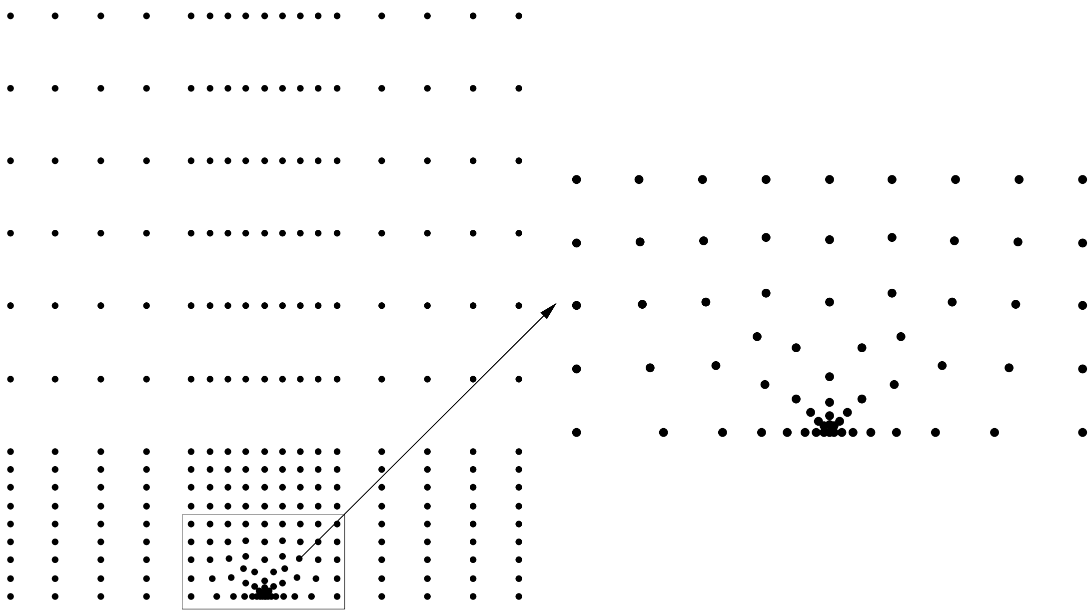
Radial and Hoop Strains as a Function of r



Radial and Hoop Stresses as a Function of r

II. Edge-Cracked Plate in Tension





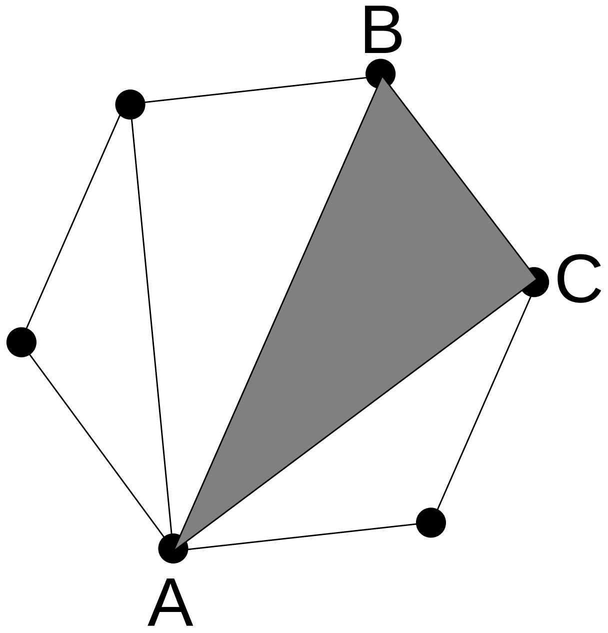
Regular (99 nodes) and Refined (278 nodes) Nodal Discretizations

Normalized K_I: Edge-Cracked Plate[a]

Nodal Grid	Domains a_1	Domains b_1	$\frac{K_I}{\sigma\sqrt{\pi a}}$ [NEM]	% E	$\frac{K_I}{\sigma\sqrt{\pi a}}$ [FE]	% E
Regular	2.8	3.0	2.52	10.7	2.16	25.1
	3.5	5.0	2.53	10.5	2.17	23.3
	3.5	8.0	2.54	10.2	2.19	22.2
Refined	1.0	2.0	2.79	1.4	2.66	6.0
	1.0	6.0	2.79	1.3	2.66	5.9
	3.5	8.0	2.79	1.4	2.68	5.3

[a] Ref: $K_I^{\mathtt{ref}}/\sigma\sqrt{\pi a} = 2.83$ (Tada, Paris, and Irwin, 1977)

C^1 Interpolants for PDEs



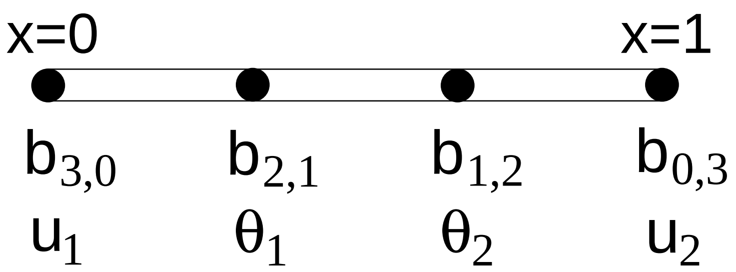
Finite Element Method

- Function as well as gradient continuity needs to be satisfied along edges AB, AC and BC.
- First- and second-order nodal degrees of freedom are in general necessary in order to construct a C^1 FEM interpolant.

Natural Element Method

- Natural neighbor coordinates are C^∞ everywhere except at the nodes where they are C^0.
- By embedding C^0 natural neighbor shape functions in the surface representation of a Bernstein-Bézier cubic simplex, a C^1 interpolant is realized (Farin, 1990).
- A transformation matrix $\mathbf{T}$ ($\mathbf{b} = \mathbf{T}\tilde{\mathbf{u}}$) which maps nodal function and gradient values to Bézier ordinates is used to construct the C^1 NEM interpolant. In 1D,

$$\boxed{u(x) = \sum_{|\mathbf{i}|=3} B^3_{\mathbf{i}}(x) b_{\mathbf{i}} = \sum_{j=1}^{4} \psi_j(x) \tilde{u}_j}$$



Mixed Formulation for NEM

Strong Form

Find $\mathbf{u} : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}^2, \ p : \bar{\Omega} \to \mathbb{R}$ such that

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{b} = 0 \quad \text{in } \Omega,$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} + \frac{p}{\lambda} = 0 \quad \text{in } \Omega,$$

$$\sigma_{ij} = -p\delta_{ij} + \bar{C}_{ijkl}\varepsilon_{kl}, \ \bar{C}_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u,$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\mathbf{t}} \quad \text{on } \Gamma_t.$$

Weak Form

Find $(\mathbf{u}^h, p^h) \in \mathbf{V}^h \times P^h$ such that

$$\bar{a}(\mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) = \langle \mathbf{b}, \mathbf{v}^h \rangle + \langle \bar{\mathbf{t}}, \mathbf{v}^h \rangle \quad \forall \ \mathbf{v}^h \in \mathbf{V}_0^h,$$

$$b(\mathbf{u}^h, q^h) - \frac{1}{\lambda}(p^h, q^h) = 0 \ \ \forall \ q^h \in P^h,$$

$$\bar{a}(\mathbf{u}^h, \mathbf{v}^h) = 2\mu \int_{\Omega^h} \boldsymbol{\varepsilon}(\mathbf{u}^h) : \boldsymbol{\varepsilon}(\mathbf{v}^h) \, d\Omega,$$

$$b(\mathbf{v}^h, p^h) = -\int_{\Omega^h} p^h \boldsymbol{\nabla} \cdot \mathbf{v}^h \, d\Omega.$$

Displacement, Pressure Interpolation

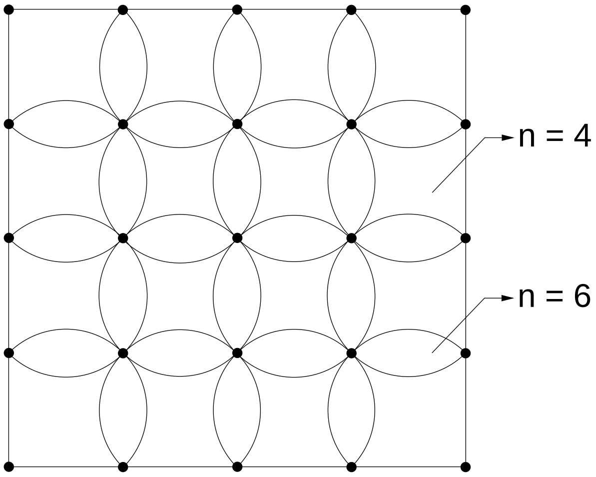
- Displacement vectors $\mathbf{u}^h(\mathbf{x}) : \ \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I=1}^{n} \phi_I(\mathbf{x}) \mathbf{u}_I$$

- Pressure field $p^h(\mathbf{x})$:

$$p^h(\mathbf{x}) = \sum_{I=1}^{n} \psi_I(\mathbf{x}) \bar{p}_I, \quad \psi_I(\mathbf{x}) = \frac{\big(\phi_I(\mathbf{x})\big)^k}{\sum_J \big(\phi_J(\mathbf{x})\big)^k}$$

$$\boxed{\begin{aligned} k = 1 &: \psi_I(\mathbf{x}) = \phi_I(\mathbf{x}) \\ k = 0 &: \psi_I(\mathbf{x}) = \frac{1}{n} \end{aligned}}$$



Discrete System for Mixed NEM

$$\begin{bmatrix} \bar{\mathbf{K}} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix},$$

$$\bar{\mathbf{K}}_{IJ} = \int_{\Omega^h} \mathbf{B}_I^T \bar{\mathbf{C}} \mathbf{B}_J \, d\Omega,$$

$$\mathbf{G}_{IJ} = -\int_{\Omega^h} \tilde{\mathbf{B}}_I^T \psi_J \, d\Omega,$$

$$\mathbf{M}_{IJ} = -\frac{1}{\lambda} \int_{\Omega^h} \psi_I \psi_J \, d\Omega,$$

$$\mathbf{f}_I = \int_{\Omega^h} \phi_I \mathbf{f} \, d\Omega + \int_{\Gamma_t^h} \phi_I \bar{\mathbf{t}} \, d\Gamma$$

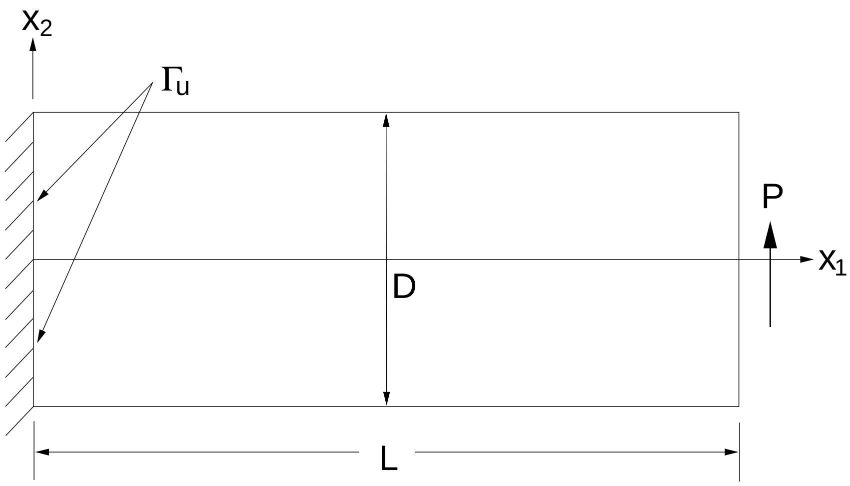
Convergence Study

- Energy error norm

$$\|\mathbf{u} - \mathbf{u}^h\|_{E(\Omega)} = \left(\frac{1}{2} a\big(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h\big) \right)^{1/2},$$

$$a(\mathbf{u}, \mathbf{u}) = 2\mu\big(\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{u})\big) - (p, \mathrm{div}\ \mathbf{u})$$

Cantilever Beam

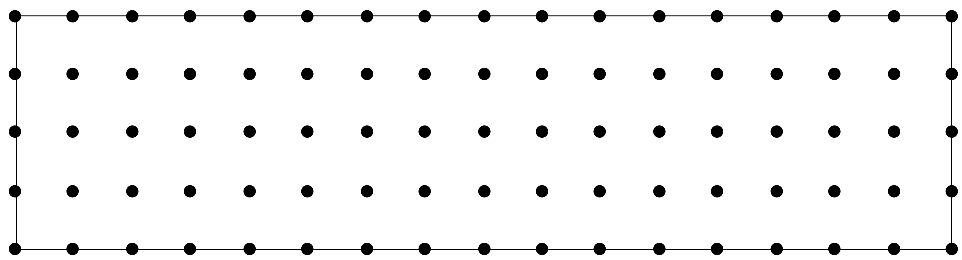


Beam Model

Material Constants and Dimensions

$$P = -1000,\ D = 1,\ L = 4$$

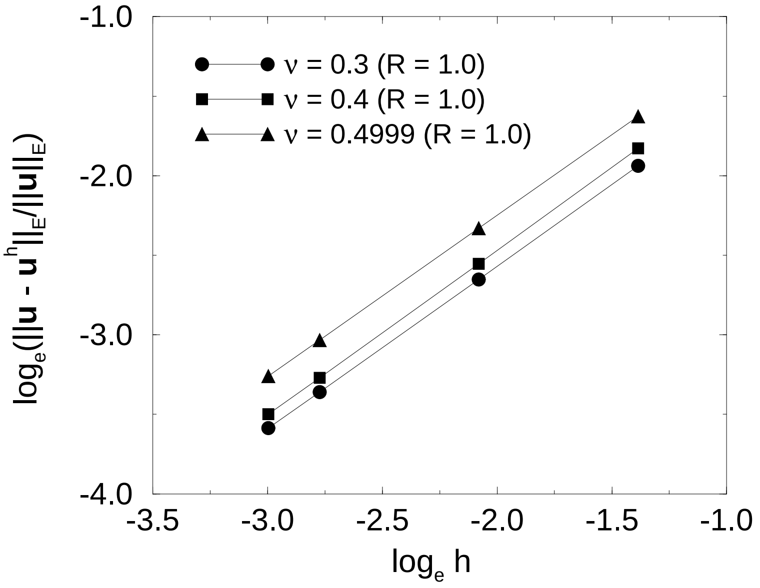
$$E = 3 \times 10^7 \text{ (plane strain conditions)}$$

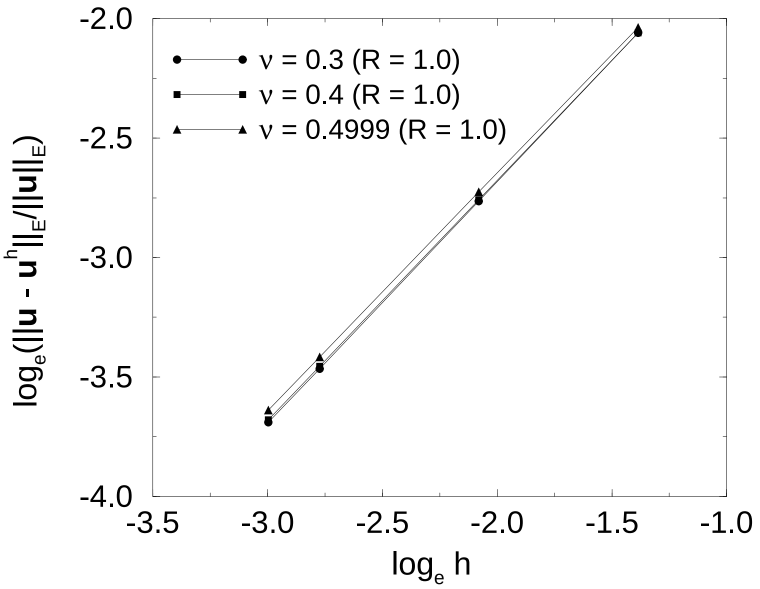


Nodal Discretization (85 nodes)

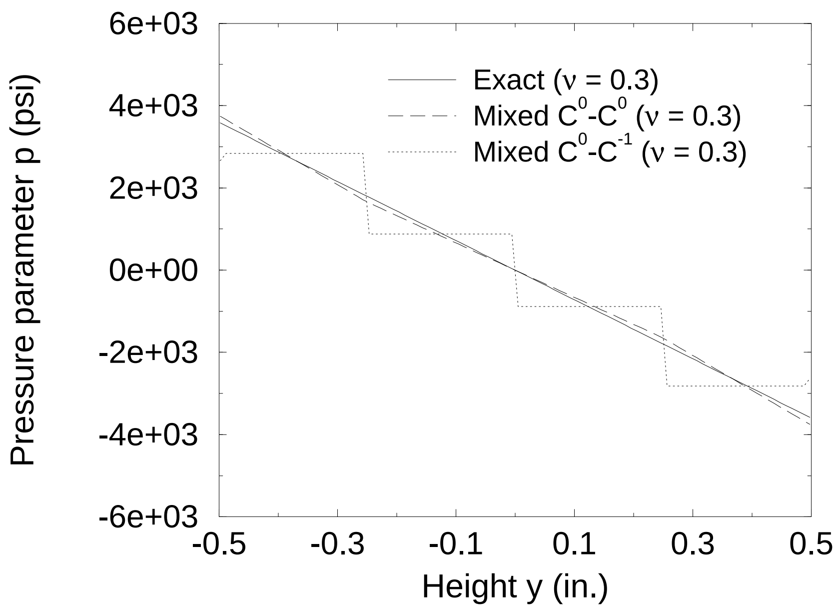
Normalized End Displacement for the Beam Problem

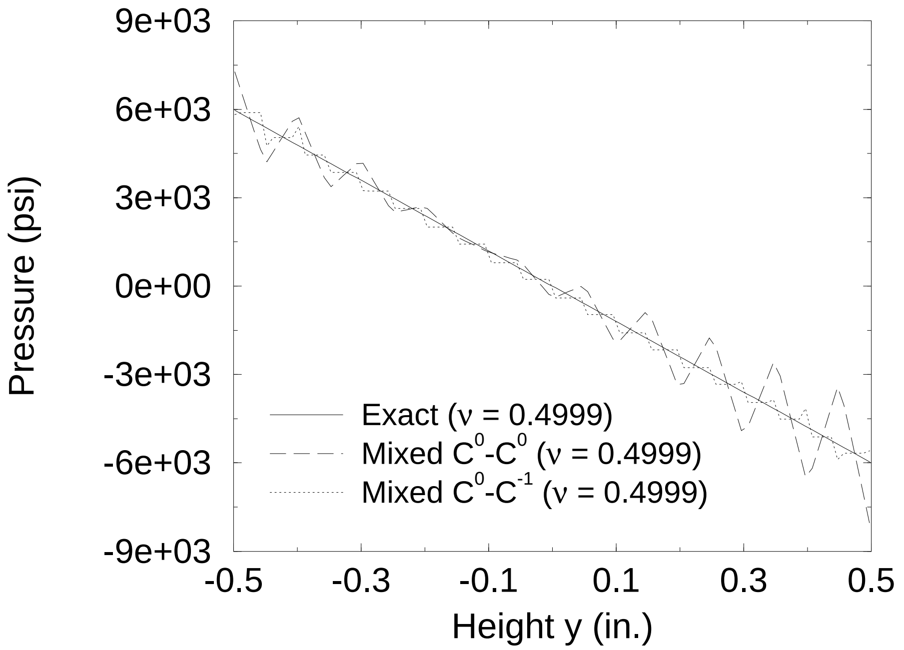
ν	FEM (3×3)	NEM C^0	NEM Mixed C^0–C^0	NEM Mixed C^0–C^{-1}
0.3	0.9651	0.9692	0.9739	0.9903
0.4	0.9373	0.9436	0.9676	0.9924
0.4999	0.1873	0.1950	0.9495	0.9901
0.4999999	0.1775	0.2103	0.9490	0.9903





Convergence in Energy. (a) Mixed C^0–C^0, and (b) Mixed C^0–C^{-1}





Variation of Pressure Parameter at $x_1 = L/2$ for the Beam Problem