Numerical Formulation and Application of Polygonal Finite Elements

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ABSTRACT: In this paper, we develop conforming Galerkin approximations on polygonal elements. A notable contribution is the use of Laplace (natural-neighbor, *nn*) basis functions on a canonical element combined with an affine map to construct conforming approximations on convex polygons. Laplace shape functions interpolate nodal data, satisfy linear completeness, and are linear on the boundary of an n-gon, which permits the direct imposition of essential boundary conditions as in classical finite element methods. We adapt this advance to quadtree meshes to obtain $C^0(\Omega)$ admissible approximations along edges with so-called *hanging nodes*. The numerical formulation with supportive numerical experiments are presented.

Key words: barycentric coordinates, natural neighbors, Laplace interpolant, convex polygons

1 INTRODUCTION

Polygonal finite elements provide greater flexibility in the meshing of complex geometries (e.g., biomechanics), are of potential use in the modeling of polycrystalline materials, useful as a transition element in finite element meshes [1], and are suitable in material design [2]. However, the development of finite elements on irregular polygons has been limited so far. Wachspress [3] proposed rational basis functions on polygonal elements, but only recently has interest in the construction of barycentric coordinates on n-gons re-surfaced [4, 5] (see [6] for details).

In this study, we present the construction of a polygonal interpolant using natural neighbor (Laplace) shape functions [7], and then use it to develop an h-adaptive finite element method on quadtree meshes. To this end, similar to three-node and four-node finite elements, the Laplace basis functions are defined on a canonical element and an affine map is used to construct conforming approximations on convex polygons as well as elements in quadtree meshes. The proposed method can be viewed as an extension of finite elements to convex n-gons ($n \geq 3$).

As an application of the proposed polygonal finite element method, we construct $C^0(\Omega)$ admissible approximations on quadtree meshes. Quadtree is a hierarchical data structure [9], which is widely used in geometric modeling and computer graphics. As a spatial data structure, efficient storage and fast data retrieval in a quadtree (or octree in 3-d) are unmatched, which renders it particularly attractive for adaptive simulations. The caveat, however, is that classical finite element methods are non-conforming (due to presence of *hanging nodes*) on quadtree meshes which limits their potential applications in computer simulation. One of the key contributions in this study is the adaptation of the natural neighbor-based (Laplace) polygonal interpolant to enable numerical computations to be performed on quadtree decompositions of any finite element mesh. In Fig. 1, quadtree data structures after one and two levels of refinements are shown. The *hanging nodes* a and b that are generated in neighboring elements at different levels of refinement are indicated.

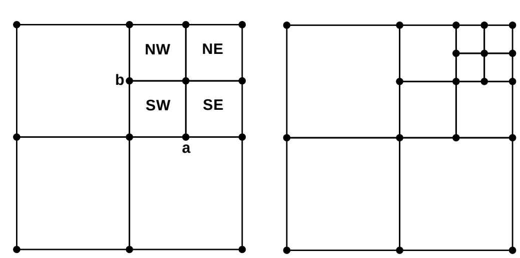

Fig. 1: Quadtree: Levels 1 (left) and 2 (right).

2 POLYGONAL INTERPOLANTS

An interpolation scheme for a scalar-valued function $u(\mathbf{x}) : \Omega \to \mathbf{R}$ is:

$$u^h(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x}) u_i, \tag{1}$$

where u_i ($i = 1, 2, \ldots, n$) are the unknowns at the n neighbors of point p, and $\phi_i(\mathbf{x})$ is the shape function for node i. From the viewpoint of a conforming Galerkin approximation, the following are the desirable properties of shape functions (barycentric coordinates) and of the interpolant: non-negativity, interpolation, partition of unity and linear completeness:

$$0 \le \phi_i(\mathbf{x}) \le 1, \quad \phi_i(\mathbf{x}_j) = \delta_{ij}, \tag{2a}$$

$$\sum_{i=1}^{n} \phi_i(\mathbf{x}) = 1, \quad \sum_{I=1}^{n} \phi_i(\boldsymbol{\xi}) \mathbf{x}_i = \mathbf{x}. \tag{2b}$$

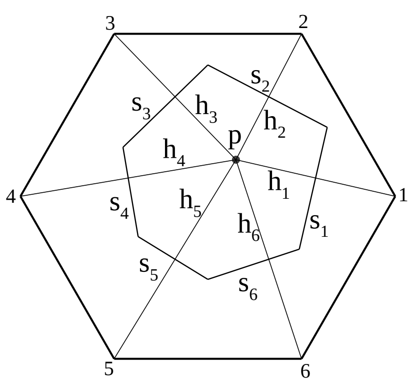

Fig. 2: Laplace shape function.

Given a set of nodes in the plane, the Laplace shape function at a point p within the convex hull is determined using the Voronoi diagram of the nodal set and p. The *natural neighbors*[10] of p are defined through the Delaunay circumcircles; if p lies within the circumcircle of a Delaunay triangle t, the nodes that define t are neighbors of p. Formally, we define the Laplace shape function as [7]:

$$\phi_i^l(\mathbf{x}) = \frac{\alpha_i(\mathbf{x})}{\sum\limits_{j=1}^{n} \alpha_j(\mathbf{x})}, \quad \alpha_j(\mathbf{x}) = \frac{s_j(\mathbf{x})}{h_j(\mathbf{x})}, \tag{3}$$

where $s_i(\mathbf{x})$ is the length of the Voronoi edge and $h_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_i\|$ (Fig. 2). The Laplace shape function satisfies all the properties indicated in Eq. (2).

In a simplex-partition of a regular polygon, all triangles have a common center and the nodes all lie on the same circumcircle. This observation underlies the construction of the proposed polygonal finite element method. In Fig. 3, the canonical elements for a triangle, square, pentagon and hexagon are shown. In each case, the nodes lie on the same circumcircle, and hence the nodes at the vertices of a polygon are the *natural neighbors* for any point in Ω_0. Since $\phi_i^l \equiv \phi_i^l(\boldsymbol{\xi})$ is piece-wise linear on the boundary $\partial\Omega_0$, we use the isoparametric mapping given in Eq. (2) from $\boldsymbol{\xi} \leftrightarrow \mathbf{x}$. Since the mapping is affine, the shape functions remain linear on the boundary of a distorted but convex polygon (Fig. 4).

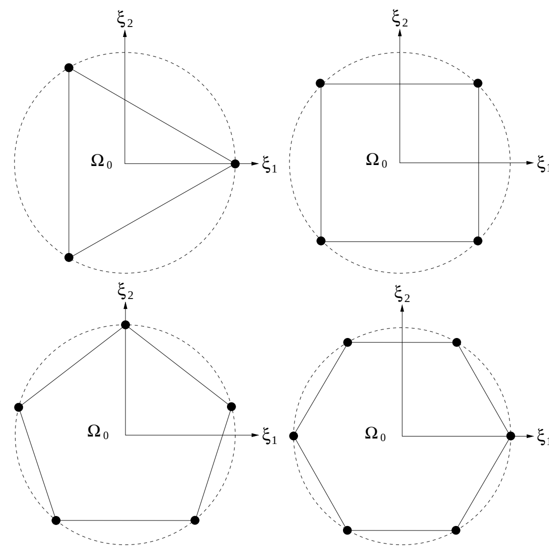

Fig. 3: Canonical elements.

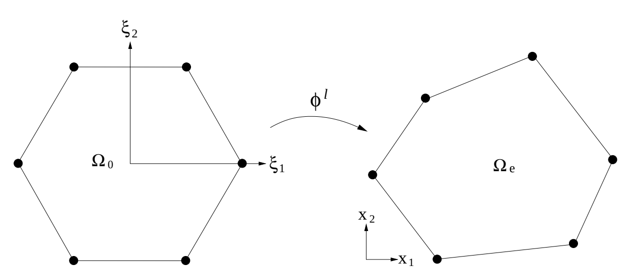

Fig. 4: Isoparametric mapping.

3 QUADTREE DATA STRUCTURE

In any spatial data structure, the domain is enclosed by unit squares (root) that are sub-divided into four equal elements (cells) which are the children of the root. This process can be repeated several times on each of the children until a stopping criteria is met. Two cells are adjacent if they have a common edge. Each child of the cell represents an element: {`NW`, `NE`, `SW`, `SE`} (Fig. 1). A cell is called a leaf if it does not have any children. The level of a cell is the number of refinements needed to obtain that cell; the root is at level zero.

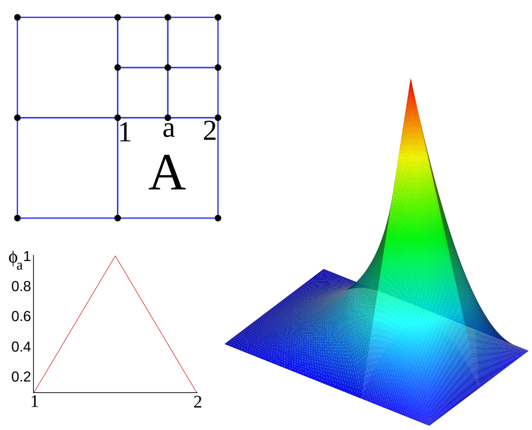

Fig. 5: Shape function ϕ_a^l.

The classical approach of using quadtree with finite elements consist of two steps: domain discretization into quadtree elements, and then sub-division of the leaf cells into finite elements [11]. The second step is required because after each refinement *hanging nodes* are generated in the adjacent elements of different level. To construct the shape functions in a quadtree element A (Fig. 5), we use the affine map from the hexagon in Fig. 3 to A. The shape function ϕ_a^l is continuous, and linear behavior along 1–a and a–2 is realized. Note that this behavior on the boundary is distinct from higher-order FEM. In this study we use the [m:1] rule: each edge can contain any number of edge nodes. Typically, in numerical computations on quadtree decompositions, the [2:1] rule (Fig. 1) is used; here, no such restrictions are imposed. For each element, we store its connectivity, refinement level, a pointer to its father, and a pointer to its children.

4 NUMERICAL EXAMPLES

Numerical examples are first presented for the patch test in $\Omega = (0,1)^2$ with $u(\mathbf{x}) = x_1 + x_2$ imposed on $\partial\Omega$. Numerical integration is performed by sub-dividing the canonical element into n triangles: the mapping $(\xi,\eta)_\triangle \rightarrow (\xi_1,\xi_2)_{\Omega_0} \rightarrow (x_1,x_2)_{\Omega_e}$ is depicted in Fig. 6. In the analyses, four different meshes are considered (Fig. 7). Relative errors in the L^2 and energy norms are: $O(10^{-6})$–$O(10^{-5})$ and $O(10^{-9})$–$O(10^{-8})$ with 25- and 13-point quadrature schemes, respectively.

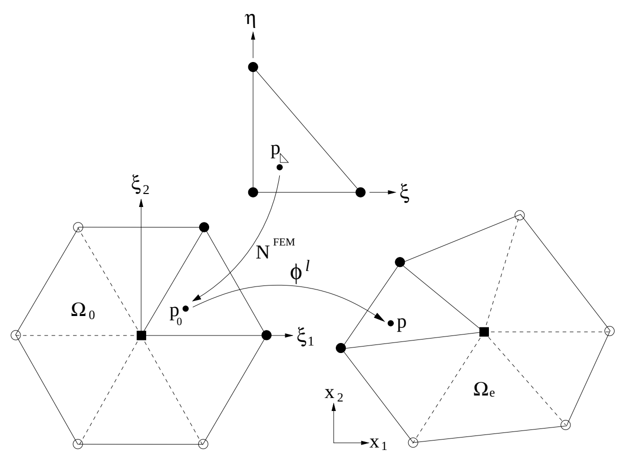

Fig. 6: Numerical integration.

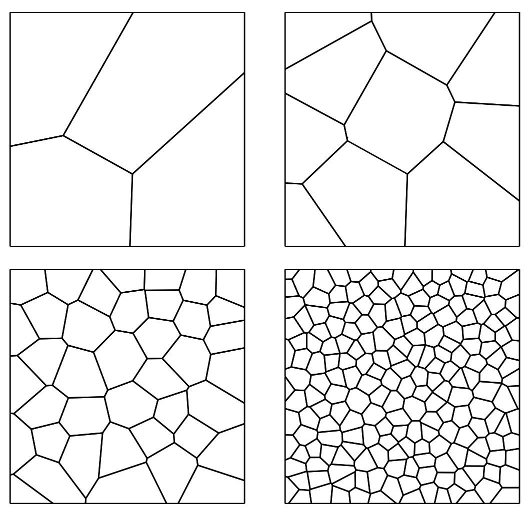

Fig. 7: Meshes used in the patch test.

For the adaptive simulations, we consider the model problem: $\nabla^2 u = f$ in $\Omega = (0,1)^2$ with $u = 0$ on $\partial\Omega$.

The source term f is chosen such that the exact solution is: $u(\mathbf{x}) = {x_1}^{10}{x_2}^{10}(1-x_1)(1-x_2)$. In Fig. 8, the initial mesh and its refinements are shown. The solution u^h on the level 5 mesh captures the sharp gradients near $(1,1)$. The relative L^2 error norm for the patch test on the meshes shown in Fig. 8 were $O(10^{-10})$.

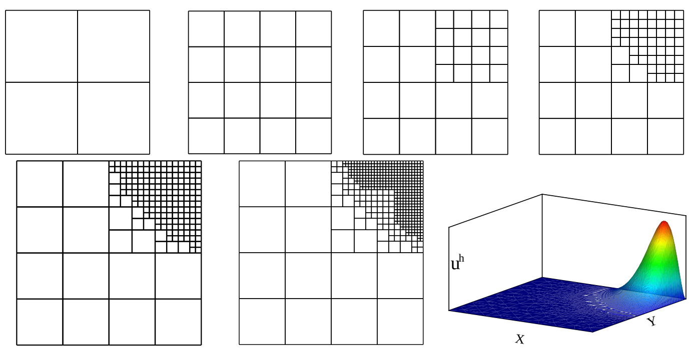

Fig. 8: Quadtree mesh refinements and u^h at level 5.

5 CONCLUDING REMARKS

In this paper, we presented the construction and numerical implementation of a conforming polygonal finite element method. The Laplace shape functions [7] were defined on a canonical (reference) element and an affine map was used to obtain the shape functions and their gradients on irregular polygons. Numerical results for the patch test were presented, and errors in the $L^2(\Omega)$ and energy norms of $\mathcal{O}(10^{-9})$ and $\mathcal{O}(10^{-8})$, respectively, were obtained. As an application of the polygonal interpolant, we developed an h-adaptive finite element method on quadtree meshes. The technique preserves continuity on edges with multiple-nodes, and avoids the need to use Lagrange multipliers or multi-point constraints, as is usually required. The simplicity of the method, both, in its construction and coding requirements, renders it distinct from some of the other recent developments on conforming approximations on quadtree meshes [12, 13].

References

[1] C. R. Dohrmann, S. W. Key, and M. W. Heinstein. A method for connecting dissimilar finite element meshes in two dimensions. *International Journal for Numerical Methods in Engineering*, 48:655–678, 2000.

[2] A. R. Diaz and A. Bénard. Designing materials with prescribed elastic properties using polygonal cells. *International Journal for Numerical Methods in Engineering*, 57(3):301–314, 2003.

[3] E. L. Wachspress. *A Rational Finite Element Basis.* Academic Press, New York, N. Y., 1975.

[4] M. Meyer, H. Lee, A. H. Barr, and M. Desbrun. Generalized barycentric coordinates for irregular n-gons. *Journal of Graphics Tools*, 7(1):13–22, 2002.

[5] M. S. Floater. Mean value coordinates. *Computer Aided Geometric Design*, 20(1):19–27, 2003.

[6] N. Sukumar and A. Tabarraei. Conforming polygonal finite elements. 2004. submitted.

[7] N. H. Christ, R. Friedberg, and T. D. Lee. Weights of links and plaquettes in a random lattice. *Nuclear Physics B*, 210(3):337–346, 1982.

[8] N. Sukumar, B. Moran, A. Yu. Semenov, and V. V. Belikov. Natural neighbor Galerkin methods. *International Journal for Numerical Methods in Engineering*, 50(1):1–27, 2001.

[9] H. Samet. The quadtree and related hierarchical data structure. *ACM Computing Surveys*, 16(2):187–260, 1984.

[10] R. Sibson. A vector identity for the Dirichlet tesselation. *Mathematical Proceedings of the Cambridge Philosophical Society*, 87:151–155, 1980.

[11] N. Provatas, N. Goldenfeld, and J. Dantzig. Adaptive mesh refinement computation of solidification microstructures using dynamic data structures. *Journal of Computational Physics*, 148(3):265–290, 1999.

[12] P. Krysl, E. Grinspun, and P. Schröder. Natural hierarchical refinement for finite element methods. *International Journal for Numerical Methods in Engineering*, 56(8):1109–1124, 2003.

[13] P. Kegan, A. Fischer, and P. Z. Bar-Yoseph. Mechanically based models: Adaptive refinement for B-spline finite element. *International Journal for Numerical Methods in Engineering*, 57:1145–1175, 2003.