Meshless Methods and Partition of Unity Finite Elements

N. Sukumar

Department of Civil and Environmental Engineering, University of California, One Shields Avenue, Davis, CA 95616. U.S.A.
URL: `http://dilbert.engr.ucdavis.edu/~suku` e-mail: `nsukumar@ucdavis.edu`

ABSTRACT: In this paper, meshless methods and partition of unity based finite element methods are reviewed. In meshless methods, the approximation is built without the explicit connectivity information between the nodes; moving-least squares approximants and natural neighbor-based interpolants are discussed. The enrichment of the finite element approximation through the partition of unity framework is described, and recent advances in crack modeling are summarized.
Key words: natural element method, extended finite element, level set method, crack discontinuities

1 INTRODUCTION

With an aim towards alleviating the need for mesh re-generation in moving boundary (such as crack growth) and large deformation problems, there has been significant interest in the development and application of meshless or meshfree methods. The impetus in this direction emanated from the work by Nayroles and co-workers who proposed the diffuse element method (DEM) [1] in 1992, and since then there have been many new developments to this class of Galerkin methods. A detailed discussion and comparison of different meshless and particle methods can be found in References [2, 3]. The meshless paradigm has provided new insights into the finite element method [4, 5], and also brought out the intimate link between scattered data approximation, computational geometry, and the numerical solution of PDEs. In particular, the partition of unity framework [4] is a powerful technique to model discontinuities and singularities through local enrichment within a finite element setting. Level set and fast marching methods (FMM) [6] are well-known interface-capturing techniques in which the interface is represented as the zero level contour of a function (level set) of one higher-dimension. The coupling of partition of unity techniques to level set methods is an appealing means to carry out geometric computations, evaluate enrichment functions (especially in 3-d), and to evolve interfaces on a fixed finite element mesh.

2 MESHLESS METHODS

Given a set of scattered nodes in $\mathbb{R}^d$ (d =1–3) with prescribed nodal data, a surface approximation can be constructed without the need for any (finite element) a priori connectedness information between the nodes. This viewpoint is adopted in meshless Galerkin methods, where well-known methods from data approximation theory [7, 8] are used to construct the trial and test spaces. We first touch upon moving least squares (MLS) approximants [7] that are used in the Element-Free Galerkin (EFG) method as well as in many of the other meshless methods [3], and then discuss natural neighbor-based interpolant schemes. In the MLS approximation, the trial function u^h for a scalar-valued function u is written as [2]

$$u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x}) a_j(\mathbf{x}) \equiv \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x})\,, \qquad (1)$$

where m is the number of terms in the basis function vector $\mathbf{p}$, and a_j are coefficients which are found by minimizing the quadratic functional J:

$$J(\mathbf{x}) = \sum_{I=1}^{n} w_I(\mathbf{x})[\mathbf{p}^T(\mathbf{x}_I)\mathbf{a}(\mathbf{x}) - u_I]^2\,, \qquad (2)$$

where $w_I(\mathbf{x}) \equiv w(\mathbf{x} - \mathbf{x}_I) \geq 0$ is a weight function with compact support. On taking the extremum of J

and after some simplification, we obtain

$$u^h(\mathbf{x}) = \sum_{I=1}^{n} \phi_I(\mathbf{x}) u_I \,, \tag{3}$$

where u_I are nodal parameters and the EFG shape functions are given by

$$\phi_I(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x}) [\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})]_{jI}. \tag{4}$$

Since $\phi_I(\mathbf{x}_J) \neq \delta_{IJ}$, the shape functions do not interpolate nodal data, which complicates the imposition of essential boundary condition in a Galerkin method. In the above equation, the matrices $\mathbf{A}$ (moment matrix) and $\mathbf{B}$ are given by

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \sum_{I=1}^{n} w_I(\mathbf{x}) \mathbf{p}(\mathbf{x}_I) \mathbf{p}^T(\mathbf{x}_I) \,, \\ \mathbf{B}(\mathbf{x}) &= [\, w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), \ldots, \; w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n) \,] \,. \end{aligned} \tag{5}$$

For smooth basis functions, the shape functions inherit the continuity of the weight function. This property provides a simple means to construct C^k $(k \geq 0)$ trial and test approximations.

In the EFG method, each node is associated with a domain of influence, which is the support of the weight function $w_I(\mathbf{x})$, with $w_I(\mathbf{x}) > 0$ in its interior and $w_I(\mathbf{x}) = 0$ outside it. Typically, domains of influence are circular or rectangular in 2-d, and Gaussian or polynomial (spline) weight functions are used [2]. The approximant used in the reproducing kernel particle method also bears close affinity to the MLS-scheme [3].

The notion of natural neighbors that was introduced by Sibson [8] is an attractive alternative to MLS approximants. The Sibson [8] and the Laplace [9] interpolants are both based on natural neighbors. The definition of natural neighbors relies on the Voronoi diagram of a nodal set. For ease of exposition, we restrict our attention to two-dimensions. The Voronoi diagram partitions a set of nodes into regions such that any point within the (first-order) Voronoi cell $\mathcal{V}(n_I)$ is closer to node n_I than to any other node. In Fig. 1, a set of seven nodes is shown, and the Voronoi diagram and its dual (Delaunay triangulation) are also indicated. A point p is introduced into the domain Ω. Now, the Voronoi diagram for p along with the seven nodes is constructed. If p and node n_I have a common Voronoi facet, then node n_I is said to be a natural neighbor of the point p [8]. In Fig. 1, the point p has five natural neighbors (filled circles).

The Sibson shape function of p with respect to a natural neighbor I is defined as the ratio of the area of the second-order Voronoi cell (A_I) to the total area A of the Voronoi cell of p:

$$\phi_I(\mathbf{x}) = \frac{A_I(\mathbf{x})}{A(\mathbf{x})}, \quad A(\mathbf{x}) = \sum_{J=1}^{n} A_J(\mathbf{x}), \tag{6}$$

where $n = 5$ and A is the polygonal (dotted line) area associated with p (Fig. 1). Let s_I be the length of the Voronoi facet, and $h_I = d(\mathbf{x}, \mathbf{x}_I)$ the distance between p and node I. The Laplace shape function for node I is defined as [9]:

$$\phi_I(\mathbf{x}) = \frac{\alpha_I(\mathbf{x})}{\sum_{J=1}^{n} \alpha_J(\mathbf{x})}, \quad \alpha_J(\mathbf{x}) = \frac{s_J(\mathbf{x})}{h_J(\mathbf{x})}. \tag{7}$$

The Sibson and Laplace shape functions are non-negative $(\phi_I \geq 0)$, interpolate nodal data, and can exactly reproduce a linear field (linearly complete) [10]. As opposed to MLS approximants, the construction of these shape functions is purely geometric with no user-defined (such as the weight function w or its support size) parameters involved in its definition, and a robust approximation is realized for non-uniform nodal discretizations in multi-dimensions. The support of shape functions based on natural neighbor and MLS-schemes is shown in Fig. 2. Consider the weak form for the Laplace equation: $\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, d\Omega = 0$. From Fig. 2, we can infer that accurate numerical integration of the weak form is an issue in meshless methods, since the intersection of shape function supports do not coincide with the integration (triangulation or quadrangulation) cells. In [11], an overview of natural neighbor-based Galerkin methods with applications in solid and fluid mechanics is presented.

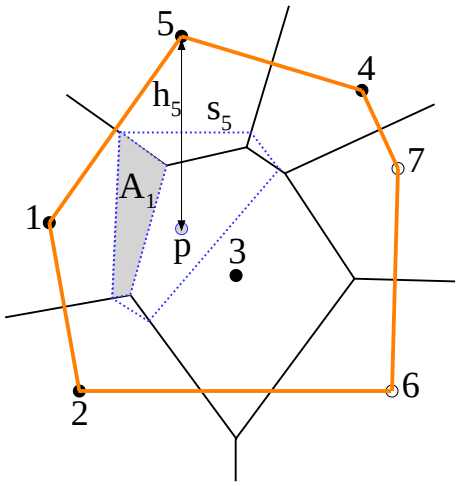


Figure 1: Sibson and Laplace shape functions.

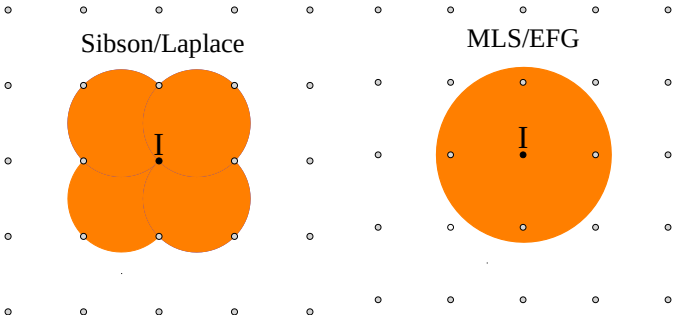


Figure 2: Support of meshless shape functions.

3 PARTITION OF UNITY FINITE ELEMENTS

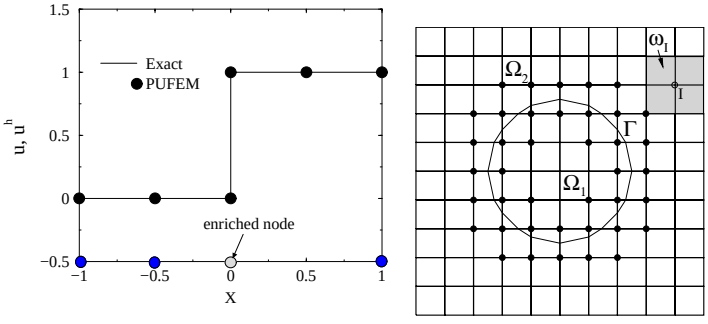
Finite element and meshless shape functions form a partition of unity (sum to unity). This realization has led to the development of partition of unity methods: in [4], the finite element shape function is used as the partition of unity, whereas in [5], the MLS approximant is adopted. The partition of unity finite element method (PUFEM) [4] is a generalization of the standard Galerkin finite element method, and possesses certain distinct advantages and improvements over previous developments in meshless methods and in p-adaptive and enriched finite elements. In finite elements, a basis function N_I is associated with node I in the mesh. Let ω_I denote the region of support for N_I: $\omega_I = \{\mathbf{x} : N_I(\mathbf{x}) > 0\}$ (see Fig. 3b). The partition of unity approximation for a scalar-valued function u can be written in the general form [4]:

$$u^h(\mathbf{x}) = \sum_{I=1}^{N} N_I(\mathbf{x}) \left(\sum_{J=1}^{M} \psi_J(\mathbf{x}) a_J^I \right), \qquad (8)$$

where $\psi_J(\mathbf{x})$ are enrichment functions, and a_J^I are unknown coefficients that are associated with: (1) node I; (2) the enrichment function ψ_J; and (3) a specific geometric entity (hole, corner, crack, or an interface). From Eq. (8), we note that the classical finite element space $\big(\psi_1 \equiv 1;\ \psi_J = 0\ (J \neq 1)\big)$ is a sub-space of the enriched space. A standard Galerkin procedure is used to obtain the discrete equations, and the symmetry and sparsity of the stiffness matrix are also retained.

The partition of unity framework satisfies a few key properties that renders it a powerful tool for local enrichment within a finite element setting: (1) can include application-specific basis functions to better approximate the solution; (2) automatic enforcement of continuity (conforming trial and test approximations); and (3) line- and surface-discontinuities can be handled without the need for the discontinuous surfaces to be aligned with the finite element mesh.

To fix ideas, we consider a simple 1-d Laplace problem in $\Omega = \Omega_1 \cup \Omega_2 = (-1, 0) \cup (0, 1)$ with a jump discontinuity $[\![u]\!] = 1$ at x = 0. The boundary conditions are such that the exact solution is: $u(x) = 0$ in Ω_1 and $u(x) = 1$ in Ω_2. We divide the domain into five nodes and enrich the middle node by a discontinuous function that is unity in Ω_2 and zero otherwise. Since a discontinuous function is embedded in the trial space, we recover the exact solution (Fig. 3a). In general, nodes whose shape function support intersect the point or surface of discontinuity need to be enriched. In 2-d, the enriched nodes for a circular interface are shown in Fig. 3b.



(a) (b)

Figure 3: Jump discontinuity. (a) 1-d; (b) 2-d.

With the above backdrop, we summarize some of the recent advances in crack modeling within the partition of unity framework. A notable use of the discontinuous function within a partition of unity method has been for 2-d and 3-d crack growth modeling without the need for any re-meshing [12, 13]; the implementation was referred to as the extended finite element method (X-FEM). For crack modeling in isotropic linear elasticity, a discontinuous function and the two-dimensional asymptotic crack-tip displacement fields are used to account for the crack, whereas in [14], the asymptotic near-tip fields for an interfacial crack are used for 2-d bimaterial interfacial crack problems. For monotonically advancing (crack) fronts, the use of a combined X-FEM/FMM method [15] holds promise for automated 3-d crack growth simulations.

4 CONCLUSIONS

We reviewed MLS- and natural neighbor-based meshless methods. The positive attributes in the latter were the ease of imposing essential boundary conditions, and the construction of robust approximations at a relatively low cost. It is simpler to construct C^k trial spaces using MLS approximants. Mathematical analysis of meshless methods in multi-dimensions is required to both, better understand its limitations, and to realize its full potential. Partition of unity methods are clearly superior when discontinuous phenomena, singularities, or small-scale features need to be captured on a coarse mesh. The partition of unity framework is particularly advantageous in 2-d and 3-d crack growth modeling, and clearly supercedes meshless methods such as EFG which have had limited success in 3-d [16].

References

[1] B. Nayroles, G. Touzot, and P. Villon. Generalizing the finite element method: Diffuse approximation and diffuse elements. *Computational Mechanics*, 10:307–318, 1992.

[2] T. Belytschko, Y. Krongauz, D. Organ, M. Fleming, and P. Krysl. Meshless methods: An overview and recent developments. *Computer Methods in Applied Mechanics and Engineering*, 139:3–47, 1996.

[3] S. Li and W. K. Liu. Meshfree and particle methods and their applications. *Applied Mechanics Review*, 55(1):1–34, 2002.

[4] J. M. Melenk and I. Babuška. The partition of unity finite element method: Basic theory and applications. *Computer Methods in Applied Mechanics and Engineering*, 139:289–314, 1996.

[5] C. A. Duarte and J. T. Oden. An *H-p* adaptive method using clouds. *Computer Methods in Applied Mechanics and Engineering*, 139:237–262, 1996.

[6] J. A. Sethian. Fast marching methods. *SIAM Review*, 41(2):199–235, 1999.

[7] P. Lancaster and K. Salkauskas. Surfaces generated by moving least squares methods. *Mathematics of Computation*, 37:141–158, 1981.

[8] R. Sibson. A vector identity for the Dirichlet tesselation. *Mathematical Proceedings of the Cambridge Philosophical Society*, 87:151–155, 1980.

[9] N. H. Christ, R. Friedberg, and T. D. Lee. Weights of links and plaquettes in a random lattice. *Nuclear Physics B*, 210(3):337–346, 1982.

[10] N. Sukumar. *The Natural Element Method in Solid Mechanics*. Ph.D. thesis, Theoretical and Applied Mechanics, Northwestern University, Evanston, IL, U.S.A., June 1998.

[11] E. Cueto, N. Sukumar, B. Calvo, J. Cegoñino, and M. Doblaré. Overview and recent advances in natural neighbor Galerkin methods. 2002. accepted for publication.

[12] N. Moës, J. Dolbow, and T. Belytschko. A finite element method for crack growth without remeshing. *International Journal for Numerical Methods in Engineering*, 46(1):131–150, 1999.

[13] N. Sukumar, N. Moës, B. Moran, and T. Belytschko. Extended finite element method for three-dimensional crack modeling. *International Journal for Numerical Methods in Engineering*, 48(11):1549–1570, 2000.

[14] N. Sukumar, Z. Huang, J.-H. Prévost, and Z. Suo. Partition of unity enrichment for bimaterial interface cracks. 2003. submitted.

[15] N. Sukumar, D. L. Chopp, and B. Moran. Extended finite element method and fast marching method for three dimensional fatigue crack propagation. *Engineering Fracture Mechanics*, 70(1):29–48, 2003.

[16] N. Sukumar, B. Moran, T. Black, and T. Belytschko. An element-free Galerkin method for three-dimensional fracture mechanics. *Computational Mechanics*, 20:170–175, 1997.