A $C^1(\Omega)$ NEM Interpolant for Fourth-Order PDEs

Internal Report

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Abstract

Farin [1] used the notion of Bézier simplices in Sibson coordinates [2] to propose a $C^1(\Omega)$ interpolant. The $C^1(\Omega)$ interpolant has quadratic precision, and reduces to a cubic polynomial between adjacent nodes on the boundary $\partial\Omega$. In this report, we present the $C^1(\Omega)$ formulation and propose a computational methodology for its implementation in the context of a Galerkin procedure for the solution of partial differential equations (PDEs). The approach involves the transformation of the original Bernstein basis functions $B^3_{\mathbf{i}}(\Phi)$ to new shape functions Ψ, such that the shape functions ψ_{3I-2}, ψ_{3I-1}, ψ_{3I} for node I are directly associated with the three nodal degrees of freedom w_I, θ_{Ix}, θ_{Iy}, respectively. Such a transformation renders the interpolant amenable to application in a Galerkin scheme for the solution of fourth-order elliptic PDEs. Numerical interpolation results have verified the smoothness and quadratic precision property of the interpolant, and currently its application to the biharmonic equation is in progress.

1 Introduction

The ability to develop $C^1(\Omega)$ interpolants over an arbitrary bounded domain Ω is a much researched and far from trivial task. In the realm of finite elements, one of the first $C^1(\Omega)$ interpolants developed was the Clough-Tocher finite element [3], and in subsequent years many of its variants have emerged [4, 5, 6, 7]. The higher-order smoothness or continuity requirement of interpolants is of interest since such class of trial functions are stipulated in a Galerkin forumulation for the solution of higher-order elliptic partial differential equations—$C^1(\Omega)$ trial functions for the biharmonic (fourth-order) equation in elasticity, with particular emphasis on thin plate Kirchhoff bending being a notable application and a case in point.

In the interest of brevity, we directly proceed to present Farin's $C^1(\Omega)$ formulation, without touching upon the foundations of Sibson's original natural neighbor interpolant [2]. For some familiarity with the Sibson interpolant and its application to second-order partial differential equations in mechanics, the interested reader can refer to [8] and [9], and the references therein.

2 $C^1(\Omega)$ NEM Interpolant

Farin [1] has proposed a $C^1(\Omega)$ interpolant based on Sibson's original $C^0(\Omega)$ natural neighbor interpolant. By embedding Sibson's coordinate in the Bernstein-Bézier representation of a cubic simplex, a $C^1(\Omega)$ interpolant is realized. Bernstein-Bézier patches and related concepts are widely used in surface approximations and in the field of computer-aided geometric design [10]. A review article on triangular Bernstein-Bézier surfaces can be found in [11], while a general treatment of multivariate polynomials over multi-dimensional simplices is given by [12].

In what follows, multi-index notation denoted by the bold characters $\mathbf{i}$ and $\mathbf{j}$ is used. Multi-indexes are tuples of non-negative integers, the components of which are subscribed starting at zero; for instance, $\mathbf{i} = (i_1, i_2, \ldots, i_n)$. The norm of a multi-index $\mathbf{i}$, denoted by $|\mathbf{i}|$, is defined to be the sum of the components of $|\mathbf{i}|$, namely $|\mathbf{i}| = i_1 + i_2 + \ldots + i_n$ [10]. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, with the property $\sum_I u_I = 1$, be the barycentric coordinate of a simplex $\delta \in \mathbf{R}^{n-1}$. A Bernstein-Bézier surface of degree m over the simplex δ can be written in the form [12]

$$b(\mathbf{u}) = \sum_{|\mathbf{i}|=m} B^m_{\mathbf{i}}(\mathbf{u}) b_{\mathbf{i}}, \tag{1}$$

where $b_{\mathbf{i}}$ is known as the Bézier ordinate associated with the control point $\mathbf{i}/m$. The control net of b is the network of $(n+1)$-dimensional points $(\mathbf{i}/m, b_{\mathbf{i}})$. In Eq. (1), $B^m_{\mathbf{i}}(\mathbf{u})$ are m-variate Bernstein polynomials in n variables. To elaborate, they are the terms in the multinomial expansion of unity, i.e.

$$(u_1 + u_2 + \ldots + u_n)^m = \sum_{|\mathbf{i}|=m} B^m_{\mathbf{i}}(\mathbf{u}), \qquad B^m_{\mathbf{i}}(\mathbf{u}) = \binom{m}{\mathbf{i}} u_1^{i_1} u_2^{i_2} \ldots u_n^{i_n}, \tag{2}$$

where $\binom{m}{\mathbf{i}}$ is the multinomial coefficient which is defined as

$$\binom{m}{\mathbf{i}} = \frac{m!}{i_1!\, i_2!\, \ldots i_n!}. \tag{3}$$

The univariate linear Bernstein polynomials are $\{1-\xi, \xi\}$, while the cubic polynomials are $\{(1-\xi)^3,\ 3(1-\xi)^2\xi,\ 3(1-\xi)\xi^2,\ \xi^3\}$, where $\xi \in [0, 1]$. Multivariate Bernstein polynomials have properties very much like their univariate counterparts. From the above equations, some of the important properties of Bernstein polynomials such as partition of unity, positivity, and cardinal interpolation, are easily inferred. The

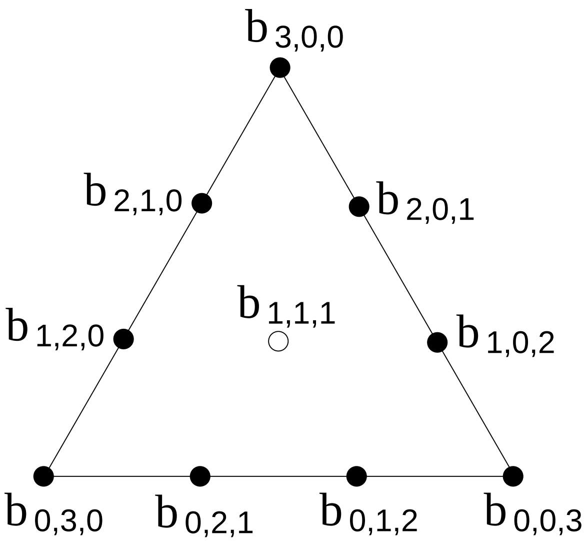

Fig. 1: Cubic Bernstein-Bézier triangular patch

control points (circles) and associated Bézier ordinate values ($b_{\mathbf{i}}$) for a cubic Bernstein-Bézier triangular patch are shown in Fig. 2. The interested reader can refer to [13], [11], and [10] for further details on the properties and applications of Bernstein-Bézier patches.

Consider a point $\mathbf{x} \in \mathbf{R}^2$ which has n natural neighbors. Let the Sibson coordinate of $\mathbf{x}$ be $\boldsymbol{\Phi} = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_n(\mathbf{x}))$. Since $\sum_I \phi_I(\mathbf{x}) = 1$, we note that $\boldsymbol{\Phi}$ can be considered as a barycentric coordinate (non-unique) of the n-gon in the plane. The generalization of Bézier surfaces over a convex polygonal domain was proposed by [14]. By using $\boldsymbol{\Phi}$ instead of $\mathbf{u}$ in Eq. (1), we can construct the surface [1]

$$w^m(\boldsymbol{\Phi}) = \sum_{|\mathbf{i}|=m} B^m_{\mathbf{i}}(\boldsymbol{\Phi}) b_{\mathbf{i}}. \tag{4}$$

In the above equation, the Bézier ordinate $b_{\mathbf{i}}$ are associated with the control point $\mathbf{q_i} \in \mathbf{R}^2$, where $\mathbf{q_i}$ are the projection of the control points of the m-variate Bézier polynomial over the $(n-1)$-dimensional simplex onto the plane [1]:

$$\mathbf{q_i} = \sum_{|\mathbf{j}|=1} B^1_{\mathbf{j}}(\mathbf{i}/m)\mathbf{x_j}, \qquad |\mathbf{i}| = m. \tag{5}$$

On the basis of Eq. (5), one can infer that the components of the barycentric coordinate $\mathbf{u}$ of the $(n-1)$-dimensional simplex is identical to that of the Sibson coordinate $\boldsymbol{\Phi}$ of the mapped n-gon on the plane.

The connectivity rule for Bézier simplexes reflects the fact that the domain simplex has all vertices connected to all other vertices. If $\mathbf{q_i}$ and $\mathbf{q_j}$ are two Bézier points in the n-gon simplex, then the rule indicates that there must exist integers r and s such that the multi-indexes $\mathbf{i}$ and $\mathbf{j}$ satisfy

$$\mathbf{i} - \mathbf{e}_r = \mathbf{j} - \mathbf{e}_s, \tag{6}$$

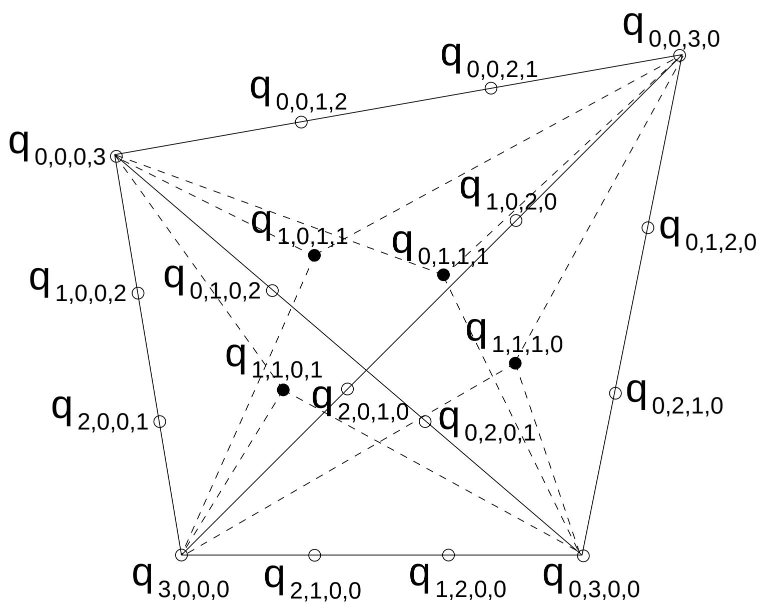

Fig. 2:Control points for the projection of a cubic tetrahedron onto the plane

where $\mathbf{e}_\alpha = (\delta_{1\alpha}, \delta_{2\alpha}, \ldots, \delta_{k\alpha}, \ldots, \delta_{n\alpha})$ denotes the multi-index having zero in all components except for the αth component, which is one. The control points for the projection of a cubic tetrahedron ($m = 3$, $n = 4$) onto the plane is shown in Fig. 2.

If we choose $m = 1$ in Eq. (4) and let w_I denote the nodal function values, we obtain

$$w^1(\mathbf{x}) = \sum_{I=1}^{n} \phi_1^{\delta_{1I}}(\mathbf{x})\phi_2^{\delta_{2I}}(\mathbf{x}) \ldots \phi_n^{\delta_{nI}}(\mathbf{x})\, b_{\delta_{1I}, \delta_{2I}, \ldots, \delta_{nI}} = \sum_{I=1}^{n} \phi_I(\mathbf{x}) w_I \tag{7}$$

which is the original Sibson interpolant. Hence, Eq. (4) can be viewed as a generalized form of the Sibson interpolant.

For $m = 3$, we arrive at the following $C^1(\Omega)$ surface representation [1]:

$$w^3(\mathbf{\Phi}) = \sum_{|\mathbf{i}|=3} B_{\mathbf{i}}^m(\mathbf{\Phi}) b_{\mathbf{i}}. \tag{8}$$

The above surface form of a cubic Bézier n-gon in Sibson coordinates is used as the $C^1(\Omega)$ NEM trial function for the solution of fourth-order PDEs.

2.1 Quadratic Precision

Farin [1] has outlined many properties of the $C^1(\Omega)$ interpolant given in Eq. (8). Here we discuss some of the important properties that are germane to a displacement-based Galerkin formulation that is pursued in this paper.

The $C^1(\Omega)$ interpolant has quadratic precison [1], i.e. it can exactly reproduce a quadratic displacement field. As opposed to the above, the $C^1(\Omega)$ interpolant proposed by [15] can only reproduce spherical quadratics, i.e. functions of the form $\alpha(x^2-y^2)$. By virtue of the quadratic precision property, the $C^1(\Omega)$ NEM interpolant can exactly represent a state of constant curvature (second-derivatives of displacement) which is required in order to pass the patch test for a fourth-order PDE such as the biharmonic equation.

By judicious choice of the Bézier ordinates, [16] realized a quadratic precision interpolant. For a cubic n-gon simplex, there are $n^2+\binom{n}{3}$ control points, and consequently the same number of Bézier ordinates. Of these, n^2 control points lie along the line joining vertices $\mathbf{q_i}$ and $\mathbf{q_j}$. The associated "boundary" Bézier ordinates to these n^2 control points have either one 3 and all other zeros (e.g. $b_{0,3,0,0}$ for $n=4$) or one 2, one 1, and other zeros (e.g. $b_{1,2,0,0}$ for $n=4$). The former (vertex or corner ordinates) are equal to the nodal function value, while the latter Bézier ordinates are easily found in the tangent planes. The additional $\binom{n}{3}$ "free" parameters are those which have three 1's and all other zeros (e.g. $b_{1,1,1,0}$ for $n=4$). An optimal choice for the center Bézier ordinate is given by $b_{1,1,1,0}=\frac{3}{2}a-\frac{1}{2}c$ [16], where a is the centroid of the tangent Bézier ordinates while c is the centroid of the vertex (corner) Bézier ordinates. The above choice of the center Bézier ordinate guarantees quadratic precision. An illustration of the calculation of the Bézier ordinates for a cubic triangular patch is shown in Fig. 2.1. Referring to Fig. 2.1, we can express $b_{1,1,1}$ as

$$b_{1,1,1,0}=\frac{3}{2}a-\frac{1}{2}c, \tag{9}$$

where

$$a=\frac{b_{2,1,0}+b_{1,2,0}+b_{2,0,1}+b_{1,0,2}+b_{0,2,1}+b_{0,1,2}}{6}, \quad c=\frac{b_{3,0,0}+b_{0,3,0}+b_{0,0,3}}{3}. \tag{10}$$

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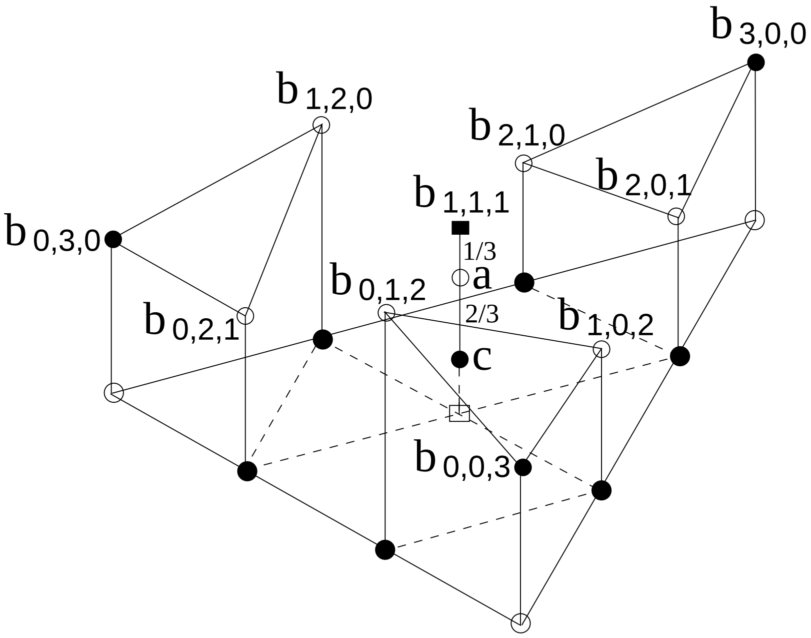

Fig. 3: Evaluation of Bézier ordinates for a cubic triangular patch

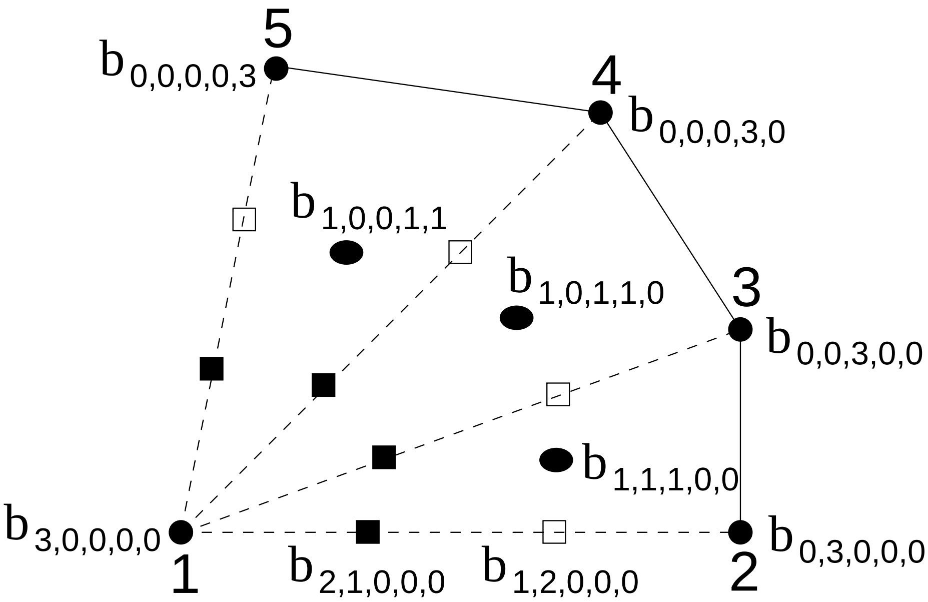

Fig. 4: Bézier ordinates used in the algorithm for a pentagonal simplex

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