Chapter 7

Conclusions

In this thesis, the application of the Natural Element Method (NEM) to elliptic boundary value problems in two-dimensional solid mechanics is explored. In NEM, the interpolants used to construct the trial and test functions are known as natural neighbor (n-n) interpolants. These interpolants are based on the Dirichlet or Voronoi tessellation of a set (N) of distinct nodes in the plane. The Voronoi tessellation is a unique and geometrically fundamental construct that defines a set of nodes. Natural neighbor interpolants have optimum spatial adjacency properties, and are sensitive to the position and the density of nodes. Natural neighbor interpolants are smooth (C^∞) everywhere, except at the nodes where they are C^0. In one-dimension, C^0 NEM interpolation is identical to linear finite elements. In the natural element method, natural neighbor interpolants are used as the trial and test functions. In two-dimensions, one way to approach the computation of natural neighbor shape functions is to calculate the areas of polygons that are obtained as the solution to a constrained linear programming problem (Lasserre, 1983). An alternate approach due to Watson (1994) for natural neighbor shape function computations was used in this thesis. The computational algorithm is based on the signed summation of areas formed by the vertices of natural neighbor circumcircles and the point $\mathbf{x}$ in the plane. A displacement-based Bubnov-Galerkin procedure was used to obtain the discrete equations in the natural element method.

The application of the C^0 natural element element to problems in two-dimensional elasticity was studied, and comparisons made to results obtained using constant strain finite elements as well as to reference solutions to validate the accuracy and convergence of the numerical method. First, an eigenanalysis was carried out to ascertain the properties of the C^0 NEM approximation spaces. Numerical results for the eigenproblem indicated that the C^0 NEM interpolant was linearly independent for both, uniform and non-uniform nodal discretizations. The condition numbers of the mass matrices using NEM were comparable to those obtained using constant strain finite elements. In the patch tests, issues pertaining to the accurate integration of the weak form were raised. Benchmark problems, such as the cantilever beam, plate with a hole, and hollow cylinder under internal pressure, which have gradients in the strain and stress solutions were solved. The numerical results were in excellent agreement with exact and analytical solutions for these problems. The ability to model material discontinuities was illustrated by solving the bi-material problem of an inclusion with a constant eigenstrain, embedded in an infinite matrix. The numerical results showed good agreement with the exact solution. The modeling of non-convex bodies, such as two-dimensional bodies with crack surfaces, was also described. The benchmark problem of an edge-crack under mode I loading was considered to test the accuracy of NEM in stress intensity factor computations. The stress intensity factors computed using NEM were within 0.3% of the reference solution, and very good domain independence was achieved.

The construction of conforming (C^k, $k > 0$) finite element spaces for $2m$ ($m > 1$) order elliptic PDEs is a non-trivial task. A desirable property of such approximation spaces in a Galerkin setting is that they interpolate nodal data for all gradients less than order m. This facilitates the imposition of essential boundary conditions by setting the appropriate nodal variables to the imposed essential boundary data. In light of the above, the C^1 natural neighbor interpolant proposed by Farin (1990b) is an attractive choice. Farin (1990b) has developed a C^1 interpolant by embedding Sibson's coordinate (Sibson, 1980) in the Bernstein-Bézier surface representation of a cubic simplex. The C^1 NEM interpolant that we proposed was constructed by a suitable transformation of Bernstein basis functions that appear in Farin's C^1 interpolant. This transformation resulted in the representation of the interpolant in terms of nodal function and nodal gradient values, which rendered it amenable to use in a Galerkin scheme for the solution of PDEs. The C^1 NEM interpolant has quadratic completeness, interpolates to nodal function and nodal gradient values, and reduces to a cubic polynomial on the boundary of the domain. In one-dimension, C^1 NEM interpolation is identical to cubic Hermite finite elements. In the application to the biharmonic equation, a standard Bubnov-Galerkin procedure was used to obtain the discrete equations. In the patch test for the biharmonic operator, issues pertaining to the accurate integration of the weak form were raised. Excellent agreement with the exact solution for the uniformly loaded clamped circular plate was obtained, and convergence to the exact center displacement was monotonic with nodal refinement. Issues pertaining to stability were probed by carrying out an eigenanalysis. Numerical results showed that the mass matrix was not well-conditioned. The satisfaction of the ellipticity condition was met, with the ellipticity constant α being the fundamental frequency of the clamped plate. This established the stability of the discrete biharmonic operator. The C^1 NEM results for the fundamental frequency matched the reference solution for the clamped plate problem. In this context, a potential application of the C^1 NEM interpolant is in the numerical computation of natural modes and frequencies of plates.

A mixed formulation for the natural element method in linear elastostatics was presented. A displacement-pressure mixed formulation was adopted with displacements interpolated by C^0 natural neighbor interpolants; C^0 and C^{-1} interpolation schemes were considered for the interpolation of the pressure. The mixed C^0-C^{-1} NEM method removed locking in the near incompressible limit ($\nu \rightarrow 0.5$) for the elastostatic boundary value problem; moreover, convergence rates in displacement and energy were optimal for all $\nu \in [0,\, 0.5)$. Numerical results for the beam in bending and for a plate with a circular hole under tension were obtained. Excellent agreement with analytical and exact solutions was obtained for problems in compressible and near incompressible media.

The computational time and costs incurred in the implementation and execution of any numerical method is a critical component in the evaluation of its feasibility, usability, and potential for mid- to large-scale applications. In the natural element method, natural neighbor coordinates are used to construct the approximation spaces. The numerical computation of the NEM interpolant is based on well-established computational geometric algorithms with optimum time-complexity. The timings involved in evaluating natural neighbor shape functions are only about twice that of constant strain finite elements. The computational costs of the C^1 natural element method are more than its C^0 variant. The increased costs are primarily due to the fact that three degrees of freedom are associated with each node; the C^1 interpolating functions require basic linear algebra matrix-vector operations and the costs are not significantly more than those incurred in the evaluation of C^0 natural neighbor interpolants. The above facts and observations do indicate that, both the C^0 and C^1 variants of NEM are very feasible and attractive choices from the computational cost and time viewpoint.

In this thesis, various applications of the natural element method were developed and applied to elliptic boundary value problems in solid mechanics. The numerical results demonstrated the capabilities, versatility, accuracy, and robustness of the numerical method, and indicate that the natural element method is a promising numerical tool in the realm of *meshless methods* to solve boundary value problems of mathematical physics.