European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS 2004
P. Neittaanmäki, T. Rossi, S. Korotov, E. Oñate, J. Périaux, and D. Knörzer (eds.)
Jyväskylä, 24–28 July 2004

POLYGONAL INTERPOLANTS: CONSTRUCTION AND ADAPTIVE COMPUTATIONS ON QUADTREE MESHES

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Key words: Meshfree Methods, Natural Neighbors, Laplace Interpolant, Barycentric Coordinates, Convex Polygons, Shannon Entropy

Abstract. *In this paper, recent advances in meshfree approximations, computational geometry, and computer graphics are described towards the development of polygonal interpolants. A particular and notable contribution is the use of meshfree (Laplace) basis functions that are based on the concept of natural neighbors. These are defined on a canonical element and an affine map is used to construct conforming approximations on convex polygons. The Laplace shape functions are non-negative, interpolates nodal data, and are linear on the boundary of the domain which permits the direct imposition of essential boundary conditions. We adapt the above construction on polygonal elements to quadtree meshes to obtain* $C^0(\Omega)$ *admissible approximations along edges with "hanging nodes." Numerical examples are presented to demonstrate the accuracy and performance of the polygonal and h-adaptive finite element methods.*

1 INTRODUCTION

Polygonal finite elements provide greater flexibility in the meshing of complex geometries (e.g., biomechanics), are of potential use in the modeling of polycrystalline materials, useful as a transition element in finite element meshes [1], and are suitable in material design [2]. However, the development of finite elements on irregular polygons has been limited so far. Wachspress [3] proposed rational basis functions on polygonal elements, but only of late has interest in the construction of barycentric coordinates on n-gons resurfaced [4, 5]. In this paper, we present a few recently proposed polygonal interpolants [4, 5], and also describe the construction of a polygonal finite element method that is based on natural neighbor (Laplace) shape functions [6]. The Laplace shape functions on polygonal elements are adapted to develop an h-adaptive finite element method on quadtree meshes. Since the determination of a linearly complete polygonal interpolant leads to an under-determined linear system for the shape functions, there exists infinite such interpolants. In [9], the least-biased interpolant or equivalently the one that maximizes the Shannon entropy [7] is derived. An extensive analysis and detailed numerical studies of the above developments appear in [8, 9, 10].

2 POLYGONAL INTERPOLANTS

An interpolation scheme for a scalar-valued function $u(\mathbf{x}) : \Omega \to \mathbf{R}$ is:

$$u^h(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x}) u_i, \tag{1}$$

where u_i ($i = 1, 2, \ldots, n$) are the unknowns at the n neighbors of point p, and $\phi_i(\mathbf{x})$ is the shape function for node i. In a Galerkin approximation for second-order PDEs, the following are the desirable properties of shape functions (barycentric coordinates on polygons) and of the interpolant: non-negativity, interpolation, partition of unity and linear completeness:

$$0 \le \phi_i(\mathbf{x}) \le 1, \quad \phi_i(\mathbf{x}_j) = \delta_{ij}, \quad \sum_{i=1}^{n} \phi_i(\mathbf{x}) = 1, \quad \sum_{I=1}^{n} \phi_i(\boldsymbol{\xi}) \mathbf{x}_i = \mathbf{x}. \tag{2}$$

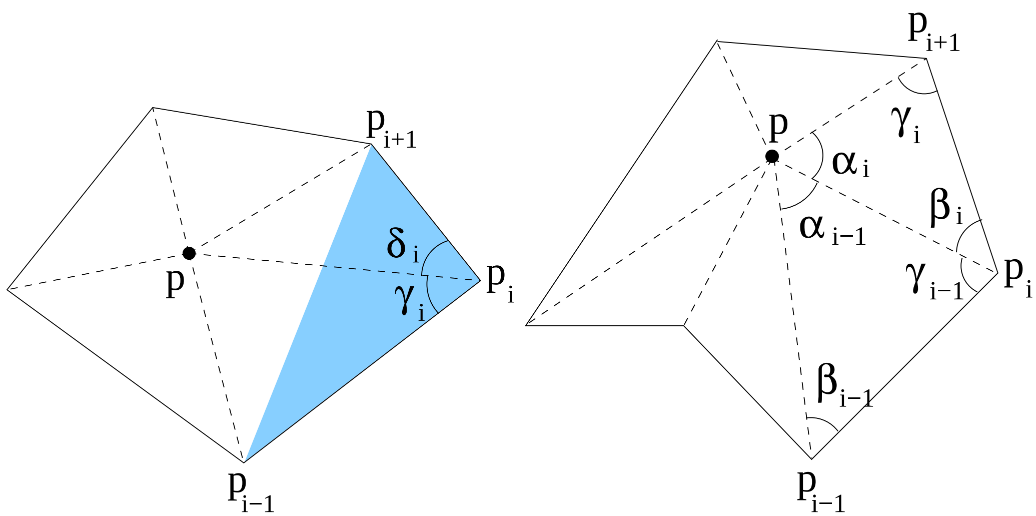
In 1975, Wachspress [3] proposed rational (ratio of polynomial functions) basis functions on polygonal elements, and renewed interest in these interpolants is a recent phenomena. Dasgupta [11, 12] used symbolic computations to compute the Wachspress basis function, whereas in [4], a simplified expression for the same is obtained:

$$\phi_i^w(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum\limits_{j=1}^{n} w_j(\mathbf{x})}, \quad w_i(\mathbf{x}) = \frac{A(p_{i-1}, p_i, p_{i+1})}{A(p_{i-1}, p_i, p) A(p_i, p_{i+1}, p)} = \frac{\cot \gamma_i + \cot \delta_i}{\|\mathbf{x} - \mathbf{x}_i\|^2}, \tag{3}$$

where the last expression is due to Meyer and co-workers, $A(a,b,c)$ is the signed area of triangle $[a,b,c]$, and γ_i and δ_i are shown in Fig. 1a. Floater [5] proposed *mean value coordinates* (barycentric coordinates for n-gons):

$$\phi_i^m(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})}, \quad w_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{\|\mathbf{x} - \mathbf{x}_i\|}, \tag{4}$$

where the angle α_i is shown in Fig. 1b.



(a)

(b)

Figure 1: Barycentric coordinates. (a) Wachspress's basis function [4]; (b) Mean value coordinates [5].

A new polygonal interpolant based on the concept of *natural neighbors* [13] was proposed in [8]. Given a set of nodes in the plane, the Laplace shape function at a point p within the convex hull is determined using the Voronoi diagram of the nodal set and p. The *natural neighbors* of p are defined through the Delaunay circumcircles; if p lies within the circumcircle of a Delaunay triangle t, the nodes that define t are neighbors of p. Formally, we define the Laplace shape function as [6]:

$$\phi_i^l(\mathbf{x}) = \frac{\alpha_i(\mathbf{x})}{\sum_{j=1}^{n} \alpha_j(\mathbf{x})}, \quad \alpha_j(\mathbf{x}) = \frac{s_j(\mathbf{x})}{h_j(\mathbf{x})}, \tag{5}$$

where $s_i(\mathbf{x})$ is the length of the Voronoi edge and $h_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_i\|$ (Fig. 2). The Laplace shape function satisfies all the properties indicated in Eq. (2) [6].

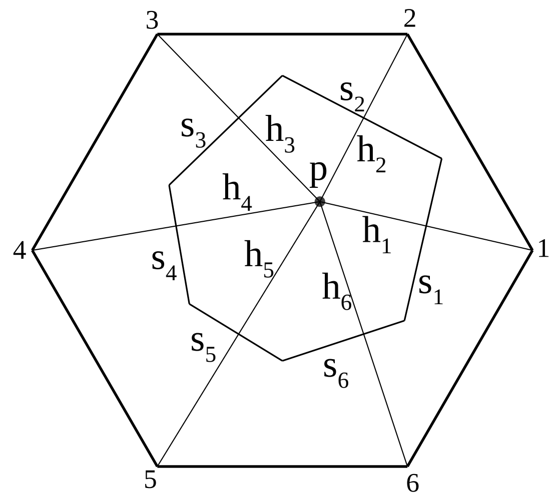


Figure 2: Laplace shape function.

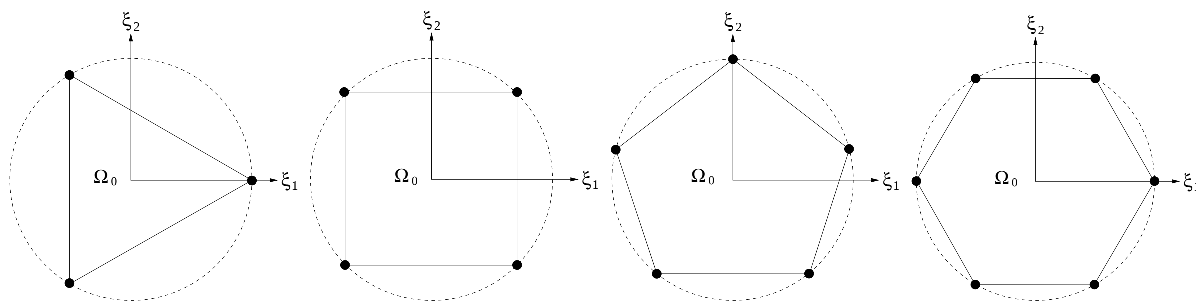


Figure 3: Canonical elements.

In a simplex-partition of a regular polygon, all triangles have a common center and the nodes all lie on the same circumcircle. In Fig. 3, the canonical elements for a triangle, square, pentagon and hexagon are shown. In each case, the nodes lie on the same circumcircle, and hence the nodes at the vertices of a polygon are the *natural neighbors* for any point in Ω_0. Since $\phi_i^l \equiv \phi_i^l(\boldsymbol{\xi})$ is piece-wise linear on the boundary $\partial\Omega_0$, we use the isoparametric mapping given in Eq. (2). Since the mapping is affine, the shape functions remain linear on the boundary of a distorted but convex polygon.

In [8], a polygonal finite element method using the Laplace shape function is presented. In the implementation, numerical integration is performed by sub-dividing the canonical element into n triangles and then numerical quadrature is performed on each triangle to assemble the stiffness matrix. We consider the patch test for the Laplace equation in $\Omega = (0,1)^2$ with $u(\mathbf{x}) = x_1 + x_2$ imposed on $\partial\Omega$. In the analyses, four different meshes are considered (Fig. 4). The relative errors in the L^2 and energy norms are: $O(10^{-6})$–$O(10^{-5})$ and $O(10^{-9})$–$O(10^{-8})$ with 25- and 13-point quadrature schemes, respectively.

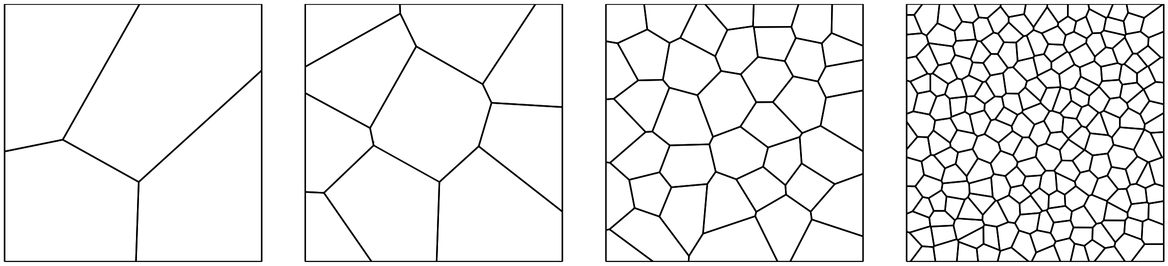
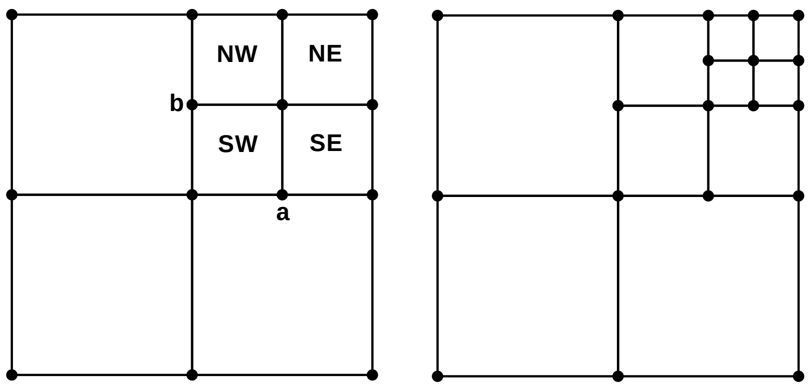


Figure 4: Meshes used in the patch test [8].



(a) (b)

Figure 5: Quadtree data structure. (a) Level 1; and (b) Level 2.

3 QUADTREE DATA STRUCTURE

Quadtree is a hierarchical data structure [14], which is widely used in geometric modeling and computer graphics. In any spatial data structure, the domain is enclosed by unit squares (root) that are sub-divided into four equal elements (cells) which are the children of the root. This process can be repeated several times on each of the children until a stopping criteria is met. Two cells are adjacent if they have a common edge. Each child of the cell represents an element: {`NW`, `NE`, `SW`, `SE`} (Fig. 5). A cell is called a leaf if it does not have any children. The level of a cell is the number of refinements needed to obtain that cell; the root is at level zero.

The classical approach of using quadtree with finite elements consist of two steps: domain discretization into quadtree elements, and then sub-division of the leaf cells into finite elements [15]. The second step is required because after each refinement *hanging nodes* are generated in the adjacent elements of different level. Recently, conforming approximations on quadtree meshes have been developed using hierarchical enrichment [16] as well as B-splines [17]. To construct the shape functions in a quadtree element A

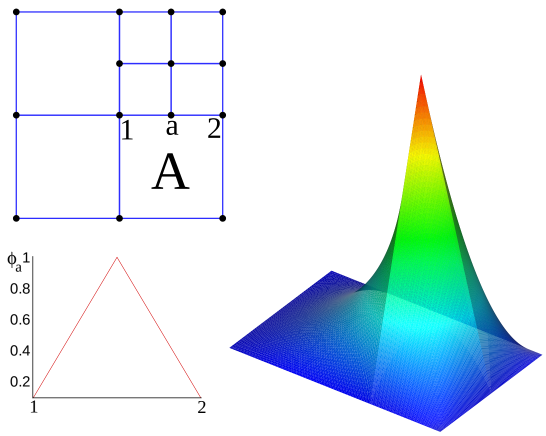


Figure 6: Shape function ϕ_a^l.

(Fig. 5a), we use the affine map from the hexagon in Fig. 3 to A. The shape function ϕ_a^l is continuous, and linear behavior along 1–a and a–2 is realized (Fig. 6). Note that this behavior on the boundary is distinct from higher-order FEM. In this study we use the [m:1] rule: each edge can contain any number of edge nodes. Typically, in numerical computations on quadtree decompositions, the [2:1] rule (Fig. 5) is used; here, no such restrictions are imposed. For each element, we store its connectivity, refinement level, a pointer to its father, and a pointer to its children. Mesh refinement strategy is based on a relative energy error norm criterion.

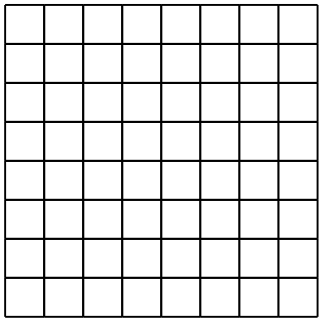
For the adaptive simulations, we consider the model problem: $\nabla^2 u = f$ in $\Omega = (0,1)^2$ with $u = 0$ on $\partial\Omega$. The source term f is chosen such that the exact solution is: $u(\mathbf{x}) = {x_1}^5 {x_2}^5 (1 - x_1)(1 - x_2)$. In Fig. 7, the initial mesh and its refinements are shown. The mesh at level 5 captures the exact solution which is depicted in Fig. 7g.

4 CONCLUDING REMARKS

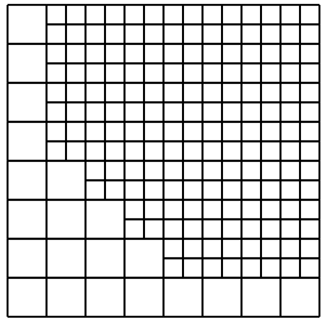
In this paper, we first reviewed some recent advances in the construction of barycentric coordinates on irregular polygons and then presented the construction of a new polygonal interpolant that was based on the concept of natural neighbors. The Laplace shape functions [6] were defined on a canonical (reference) element and an affine map was used to obtain the shape functions and their gradients on irregular polygons. The polygonal finite element method [8] can be viewed as a generalization of finite elements to convex polygons of arbitrary order. As an application of the polygonal interpolant, we developed a conforming h-adaptive finite element method on quadtree meshes [10].

5 ACKNOWLEDGEMENTS

The financial support of this work by the National Science Foundation through research award CMS–0352654 to the University of California, Davis, is gratefully acknowledged.



(a)



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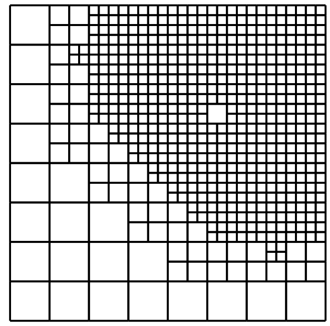

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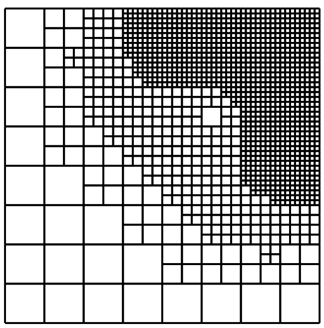
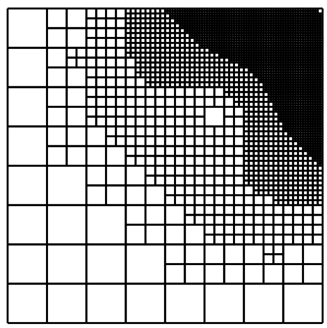


(d)



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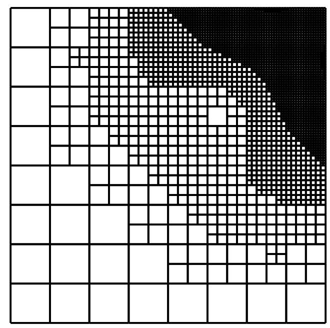

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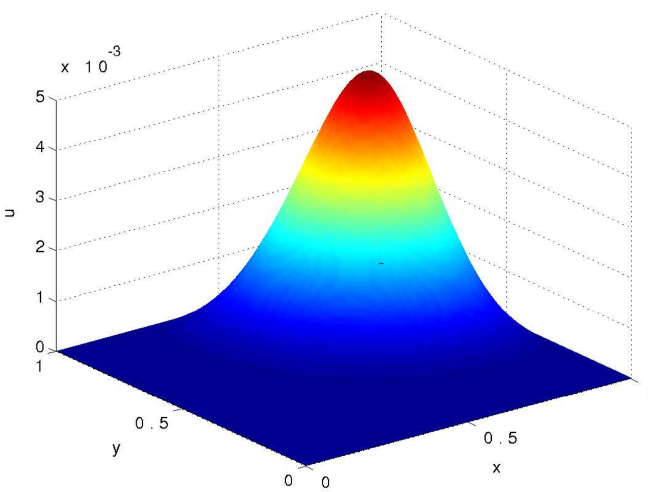

(g)

Figure 7: Mesh refinements for the Poisson problem.

REFERENCES

[1] C. R. Dohrmann, S. W. Key, and M. W. Heinstein. A method for connecting dissimilar finite element meshes in two dimensions. *International Journal for Numerical Methods in Engineering*, 48:655–678, 2000.

[2] A. R. Diaz and A. Bénard. Designing materials with prescribed elastic properties using polygonal cells. *International Journal for Numerical Methods in Engineering*, 57(3):301–314, 2003.

[3] E. L. Wachspress. *A Rational Finite Element Basis*. Academic Press, New York, N. Y., 1975.

[4] M. Meyer, H. Lee, A. H. Barr, and M. Desbrun. Generalized barycentric coordinates for irregular n-gons. *Journal of Graphics Tools*, 7(1):13–22, 2002.

[5] M. S. Floater. Mean value coordinates. *Computer Aided Geometric Design*, 20(1):19–27, 2003.

[6] N. H. Christ, R. Friedberg, and T. D. Lee. Weights of links and plaquettes in a random lattice. *Nuclear Physics B*, 210(3):337–346, 1982.

[7] E. T. Jaynes. Information theory and statistical mechanics. *Physical Review*, 106(4):620–630, 1957.

[8] N. Sukumar and A. Tabarraei. Conforming polygonal finite elements. *International Journal for Numerical Methods in Engineering*, 2004. submitted.

[9] N. Sukumar. Construction of polygonal interpolants: A maximum entropy approach. *International Journal for Numerical Methods in Engineering*, 2004. submitted.

[10] A. Tabarraei and N. Sukumar. Adaptive computations on conforming quadtree meshes. *Finite Elements in Analysis and Design*, to be submitted 2004. Special Issue on Papers Presented at the Robert J. Melosh Finite Element Medal Competition.

[11] G. Dasgupta. Interpolants within convex polygons: Wachspress' shape functions. *Journal of Aerospace Engineering*, 16(1):1–8, 2003.

[12] G. Dasgupta. Integration within polygonal finite elements. *Journal of Aerospace Engineering*, 16(1):9–18, 2003.

[13] R. Sibson. A vector identity for the Dirichlet tesselation. *Mathematical Proceedings of the Cambridge Philosophical Society*, 87:151–155, 1980.

[14] H. Samet. The quadtree and related hierarchical data structure. *ACM Computing Surveys*, 16(2):187–260, 1984.

[15] N. Provatas, N. Goldenfeld, and J. Dantzig. Adaptive mesh refinement computation of solidification microstructures using dynamic data structures. *Journal of Computational Physics*, 148(3):265–290, 1999.

[16] P. Krysl, E. Grinspun, and P. Schröder. Natural hierarchical refinement for finite element methods. *International Journal for Numerical Methods in Engineering*, 56(8):1109–1124, 2003.

[17] P. Kegan, A. Fischer, and P. Z. Bar-Yoseph. Mechanically based models: Adaptive refinement for B-spline finite element. *International Journal for Numerical Methods in Engineering*, 57:1145–1175, 2003.