Maximum Entropy Approximation

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- Professor Yueyue Fan for helpful discussions on generalized inverse
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Outline

- Motivation and Objectives
- Meshfree Approximations
- Construction of Polygonal Interpolants
- Principle of Maximum Uncertainty/Entropy
- Numerical Results
- Concluding Remarks

Motivation

Inputs (Parameters)	Simulator (Function)	Output (Response)
$\mathbf{x} \equiv (x_1, x_2, \ldots, x_d)$	$u : \mathbf{R}^d \to \mathbf{R}$	y

Seek u^h s.t.

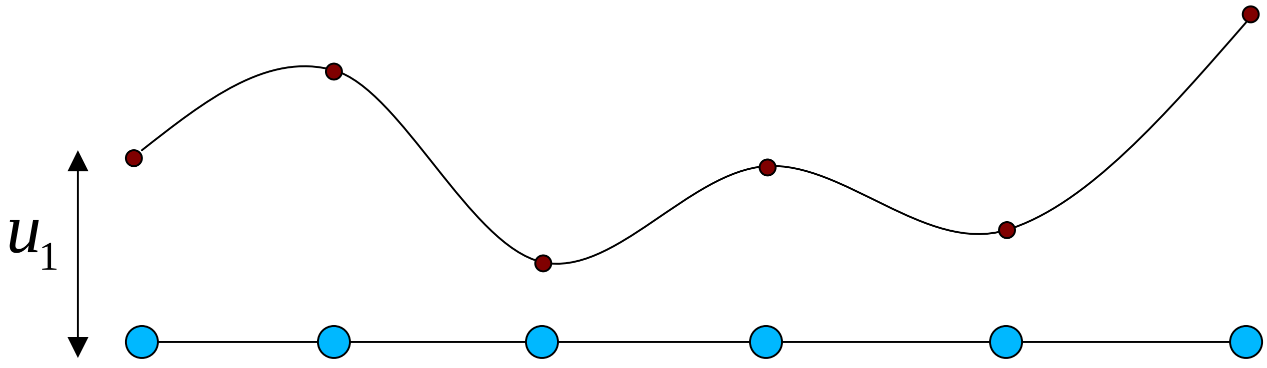
$$u^h(\mathbf{x}_i) = y_i$$

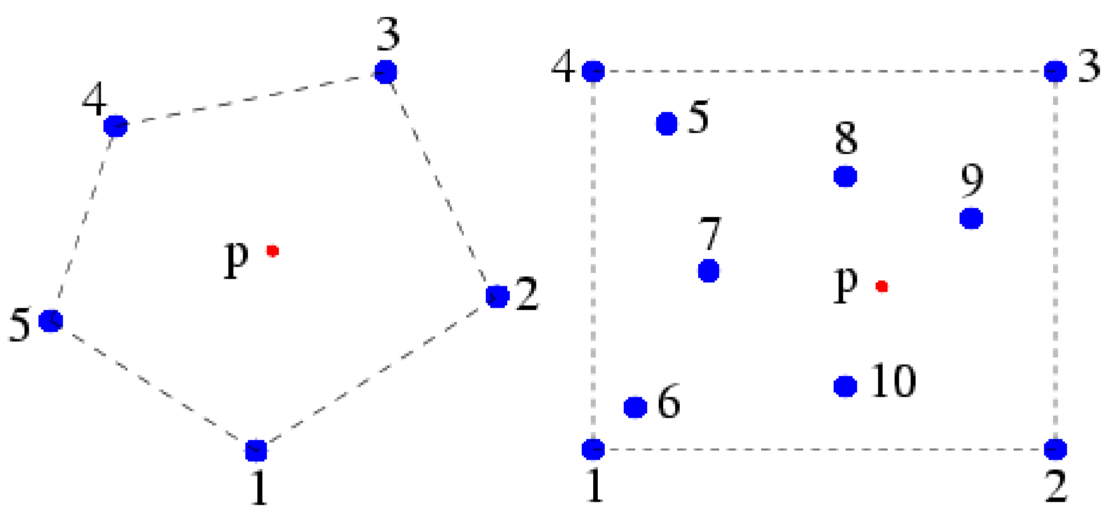
- Polynomial interpolants
- Splines
- Kriging
- Radial basis functions

Motivation: Applications

- Geometric Modeling, Computer Graphics, and Visualization
- Finite Element/Meshfree Galerkin Methods
- Numerical Estimation and Prediction
- Design and Analysis of Computer Experiments

Motivation: Data Approximation





$$u^h(\mathbf{x}) = \sum_{i=1}^{M} \phi_i(\mathbf{x})\, u_i$$

Objectives

- Merits of constructing data approximants via a constrained optimization problem
- Introduce the Maximum Entropy Principle, and to present its derivation and implementation for one-dimensional and polygonal approximation
- The promise and potential of **MAXENT** to solve problems with epistemic (ignorance) uncertainty

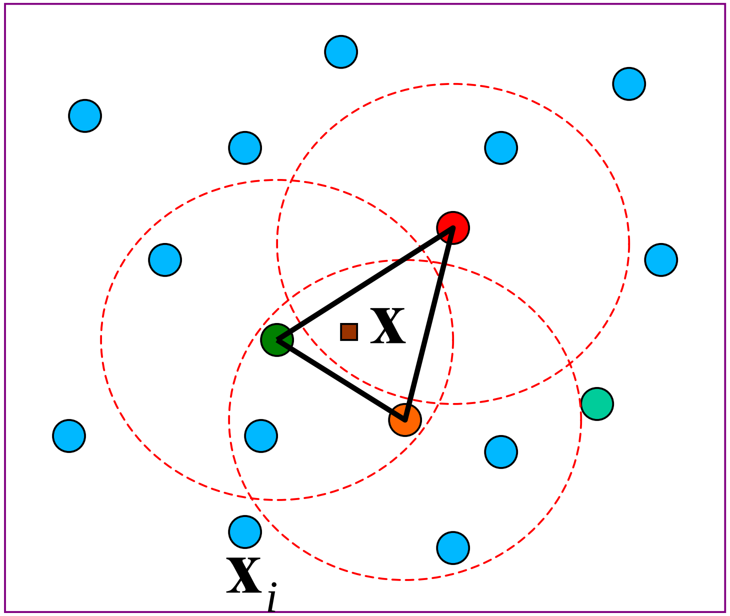
Meshfree Approximations

- DEM (Nayroles et al, 1992)
- EFG (Belytschko et al, 1994)
- RKPM (Liu et al, 1994)
- PUM (Babuska and Melenk, 1996)
- Hp-Clouds (Duarte and Oden, 1996)
- MLPG (Atluri et al, 1997)
- BNM (Mukherjee et al, 1997)
- Finite Spheres (De and Bathe, 2000)

- NEM (Braun and Sambridge, 1996)
- NEM [Laplace] (Sukumar et al, 2000)

Construction of Basis Functions

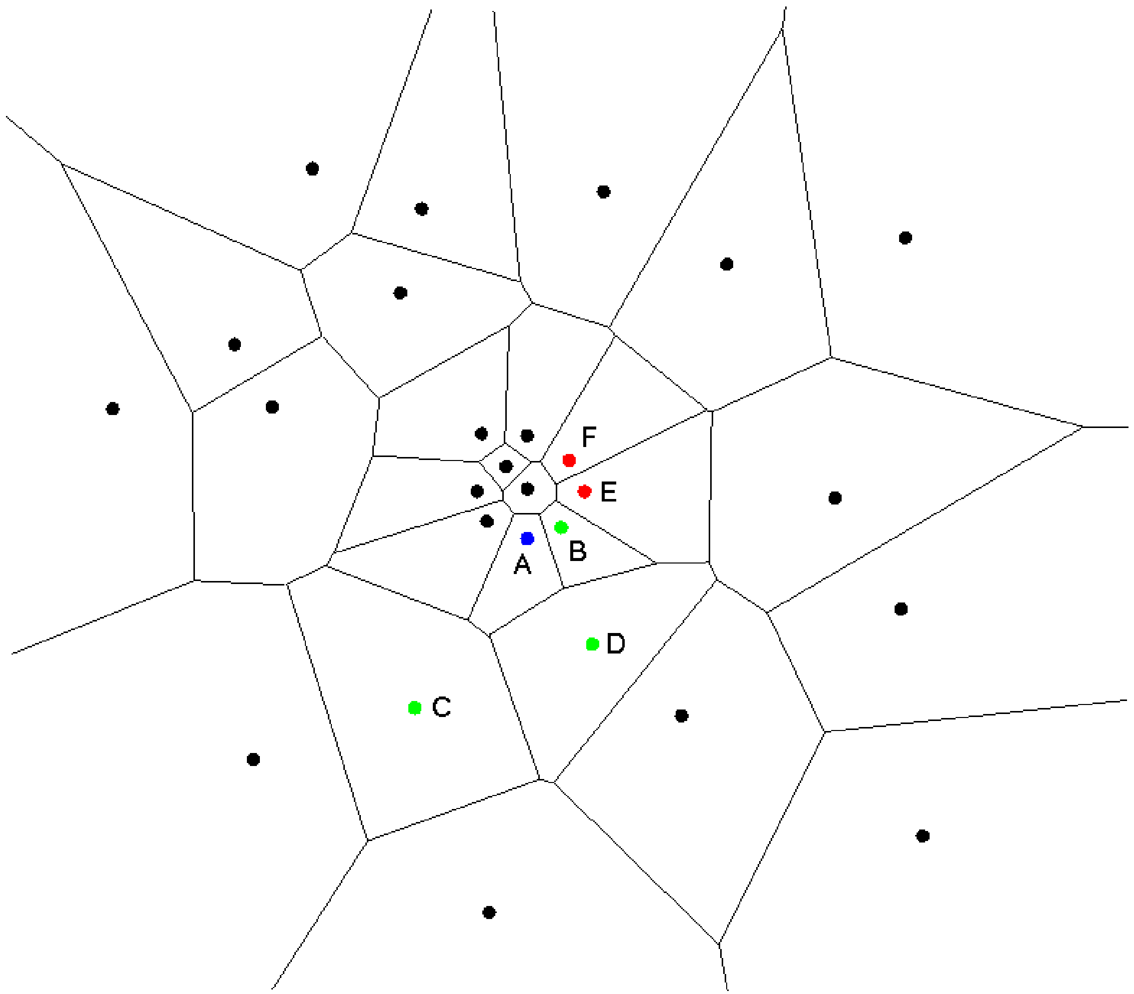
- Finite Elements
- MLS/RBFs (L^2 metric)
- Natural Neighbors
- MAXENT



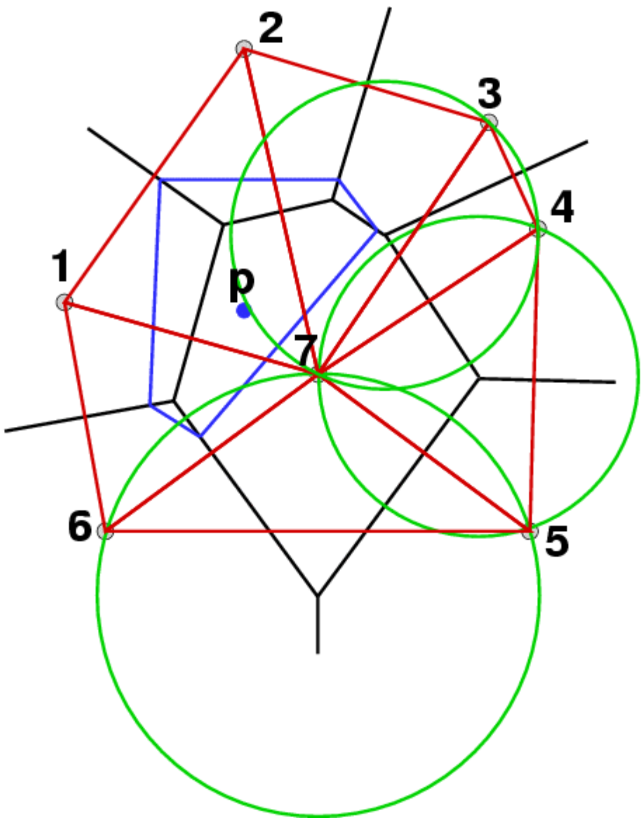
ISSUES

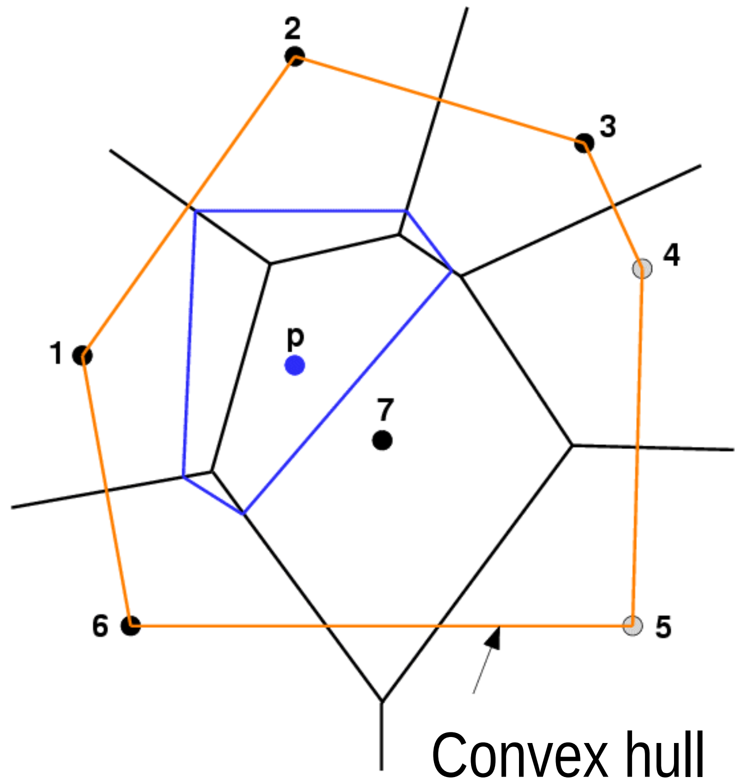
- Defining a good neighborhood: pattern recognition, clustering, learning theory
- EBCs: Interpolants are desirable
- Numerical integration (Galerkin method)

Voronoi Neighbors



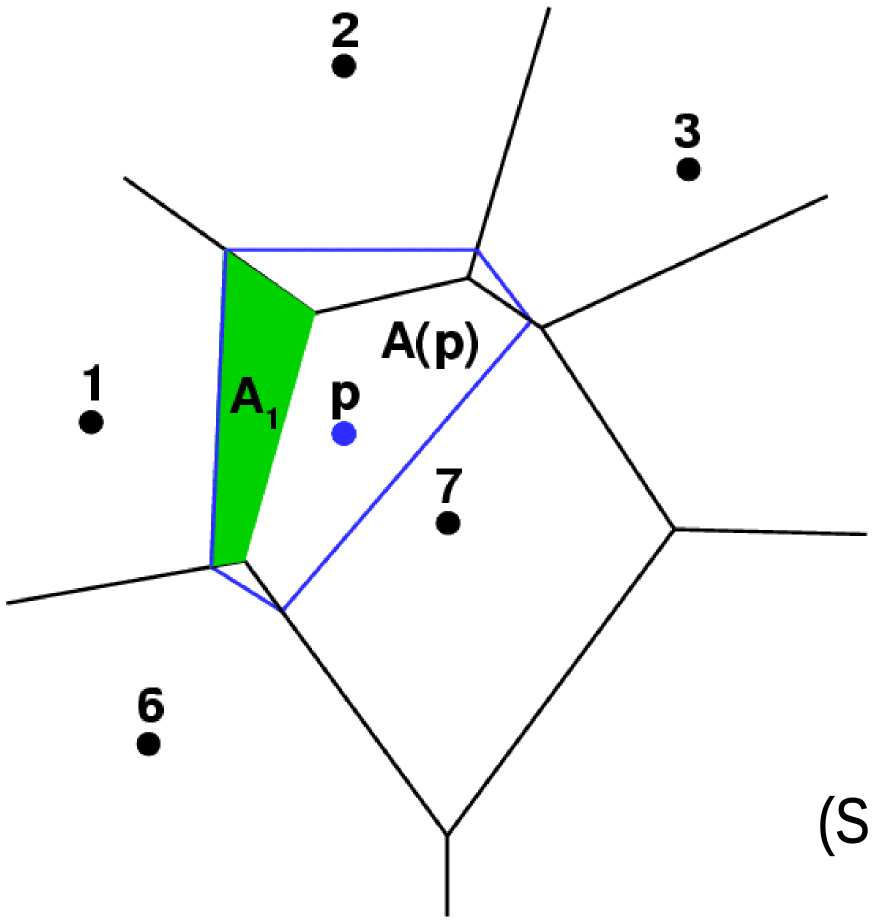
Delaunay Circumcircle and Natural Neighbors





p lies outside the circumcircles in green

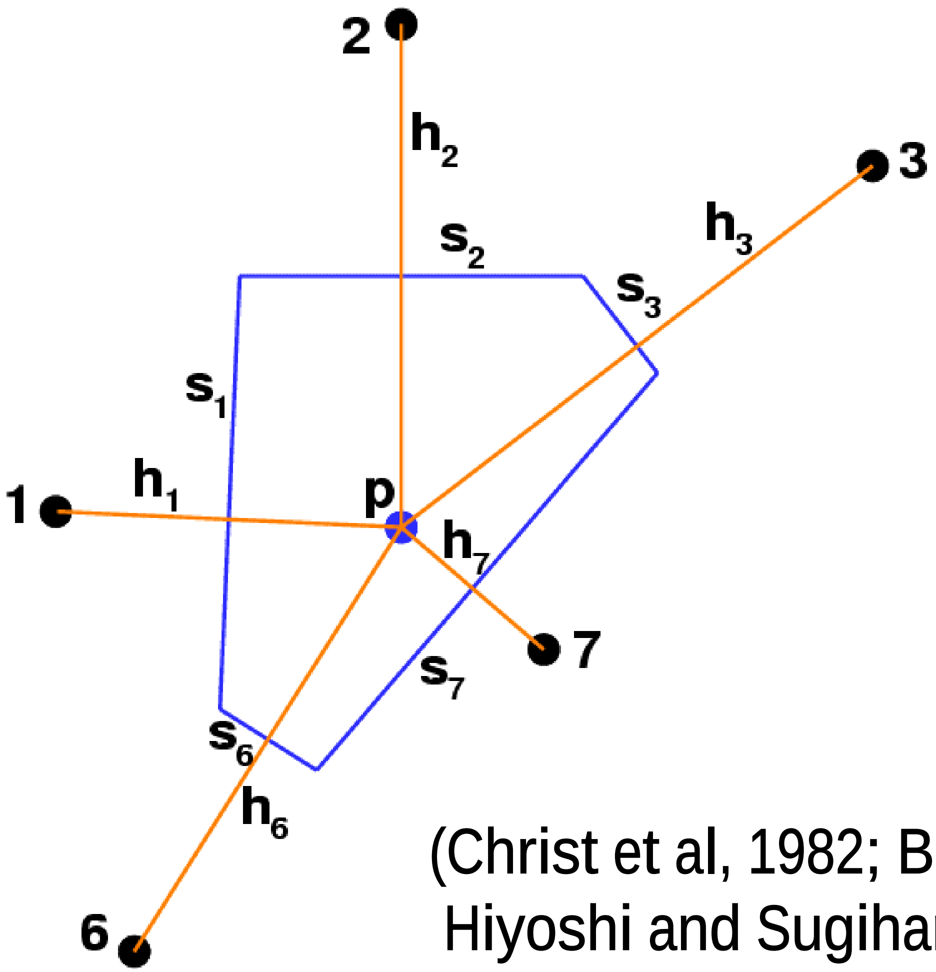
Sibson Interpolant



$$\phi_i(p) = \frac{A_i(p)}{A(p)}$$

(Sibson, 1980)

Laplace Interpolant



$$\alpha_i(p) = \frac{s_i(p)}{h_i(p)}$$

$$\phi_i(p) = \frac{\alpha_i(p)}{\sum_j \alpha_j(p)}$$

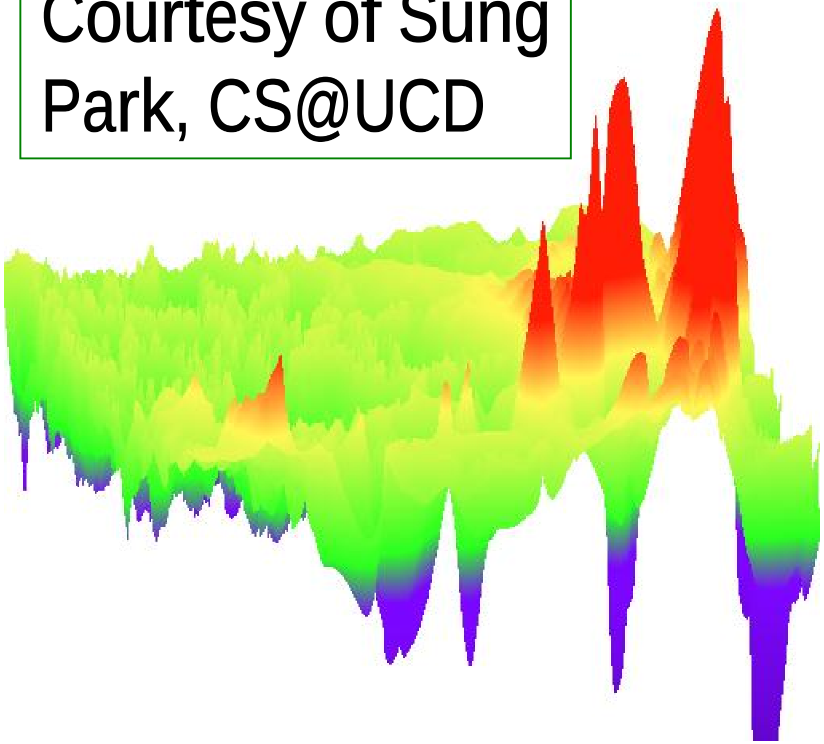
(Christ et al, 1982; Belikov et al, 1997;
Hiyoshi and Sugihara, 1999)

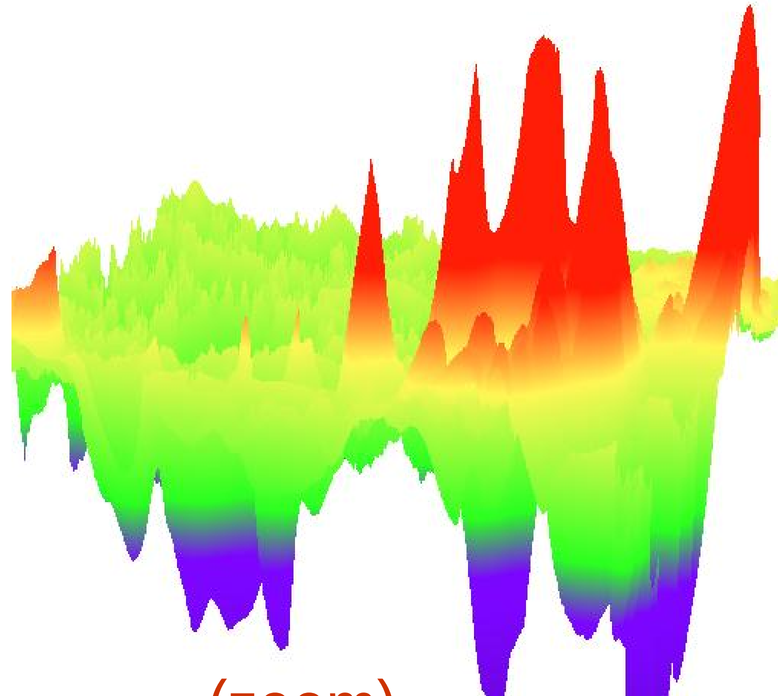
Properties

- Non-negative and PU: $0 \le \phi_i \le 1, \ \sum_i \phi_i(\mathbf{x}) = 1$
- Interpolate data: $\phi_i(\mathbf{x}_j) = \delta_{ij}$
- Linear completeness/precision: $\sum_i \phi_i \mathbf{x}_i = \mathbf{x}$
- Smoothness: $\phi_i^{\mathrm{LAP}} \in C^0(\Omega), \ \phi_i^{\mathrm{S}} \in C^1(\Omega \setminus \mathbf{x}_j)$
- Linear essential boundary conditions can be exactly imposed

Surface Interpolation (Sibson)

Courtesy of Sung Park, CS@UCD

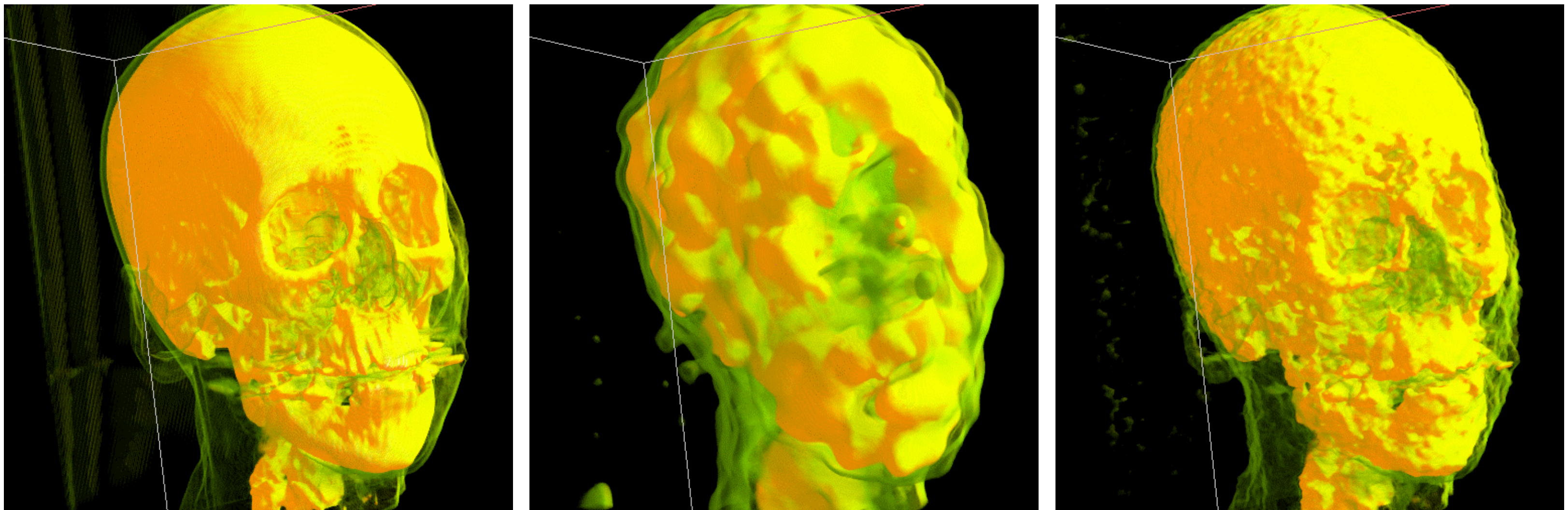




(zoom)

Bathymetry and topography data (~10,000 points) near NW Australia (Courtesy of Malcolm Sambridge)

Volume Reconstruction (Sibson): Human Head



256^3 | 10^4 | 5×10^5

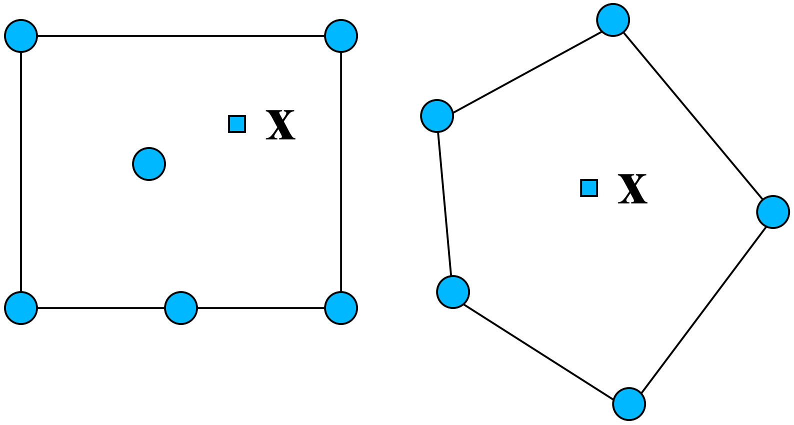
(CT scan courtesy of NC Memorial Hospital)

Courtesy of Sung Park, CS@UCD

Construction of Polygonal Interpolants

- Wachspress basis functions **(Wachspress, 1975; Warren, ACM, 1996; Meyer et al, JGT, 2002; Dasgupta, JAE, 2003; Malsch, Ph.D. thesis, 2003)**

- Mean value coordinates **(Floater, CAGD, 2003)**



- Laplace shape functions **(Sukumar and Tabarraei, IJNME, 2004)**

Construction of Polygonal Interpolants (Cont'd)

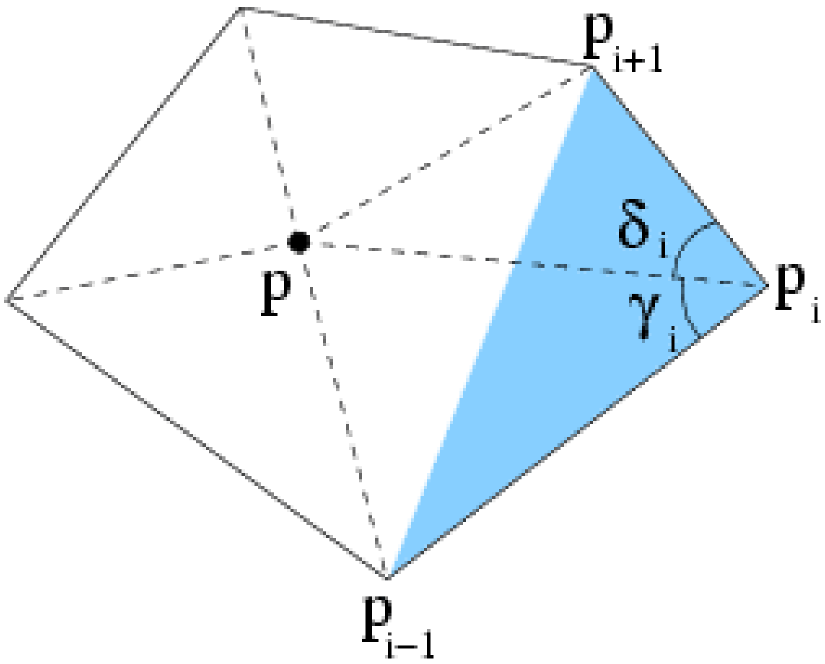
- Maximum entropy (MAXENT) shape functions **(Sukumar, IJNME, 2004)**

  - ✓ Imposing linear reproducibility leads to an under-determined system of linear equations for $\{\phi_i\}$

  - ✓ Use Shannon entropy **(Shannon, 1948)** and max entropy principle **(Jaynes, 1957)** to find $\{\phi_i\}$

  - ✓ Constrained optimization problem is solved

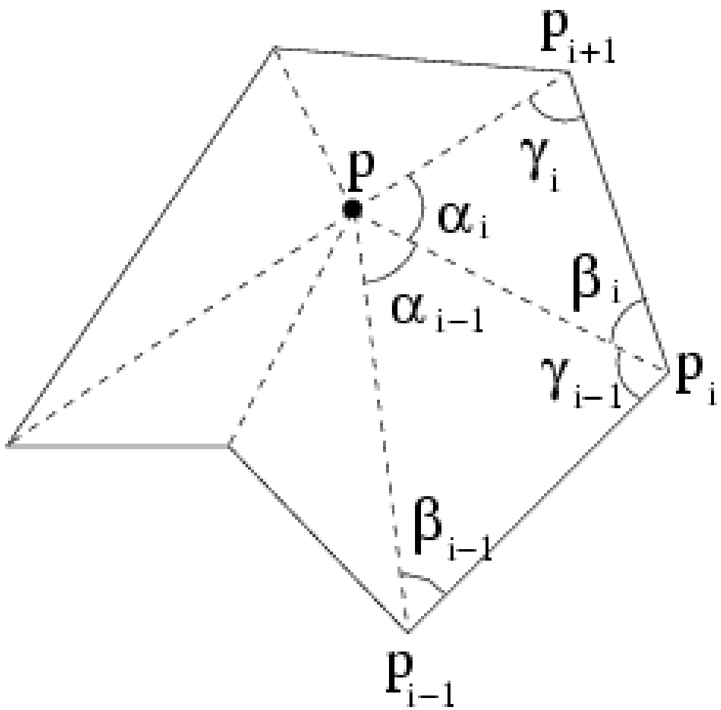
Wachspress Basis Functions



(Meyer et al., JGT, 2002)

$$\phi_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})}, \quad w_i(\mathbf{x}) = \frac{\cot \gamma_i + \cot \delta_i}{\left\| \mathbf{x} - \mathbf{x}_i \right\|^2}$$

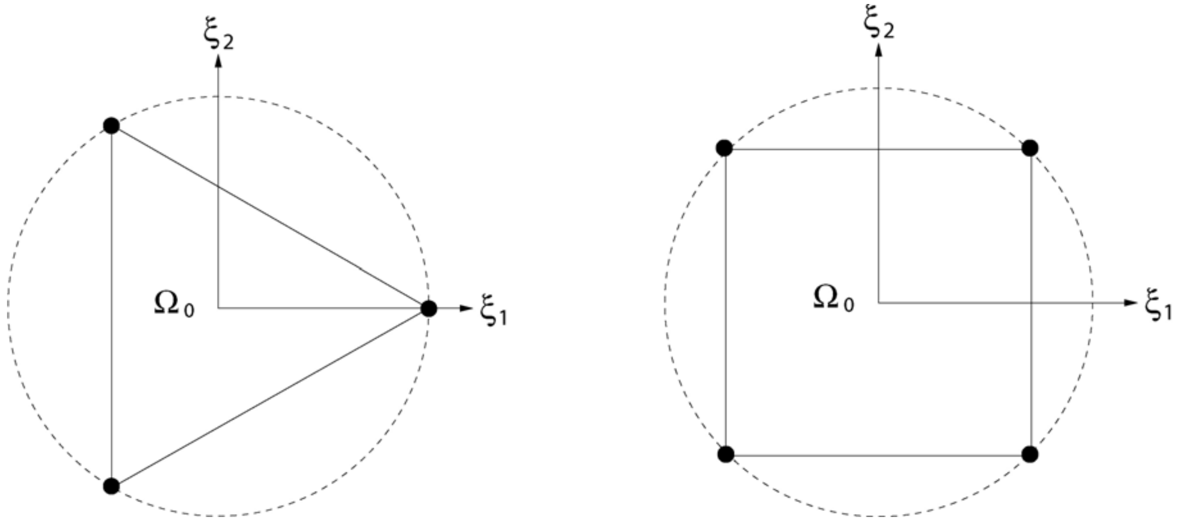
Mean Value Coordinates



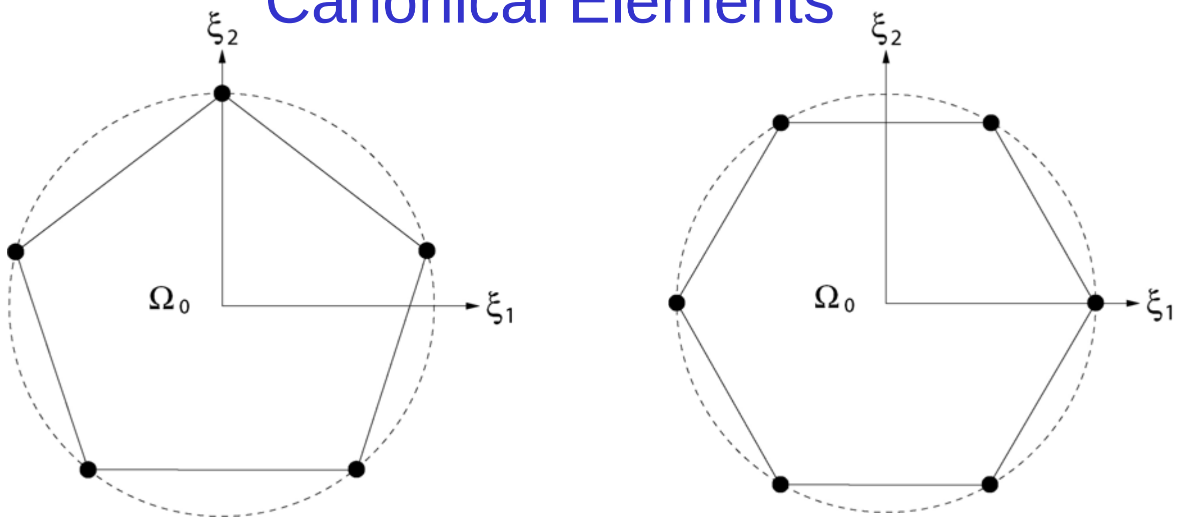
(Floater, CAGD, 2003)

$$\phi_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})}, \; w_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{\left\| \mathbf{x} - \mathbf{x}_i \right\|}$$

Laplace Shape Functions

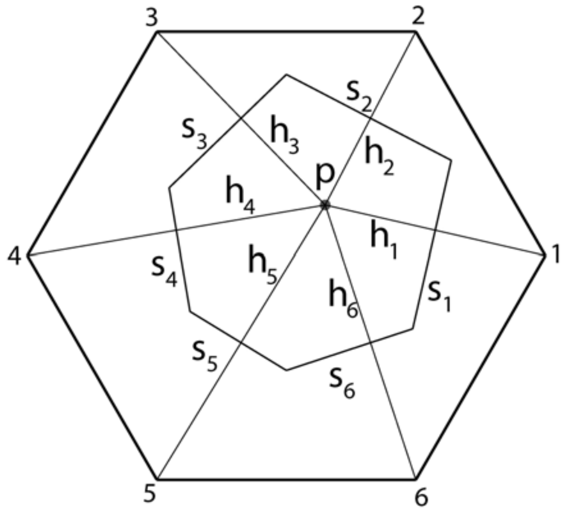


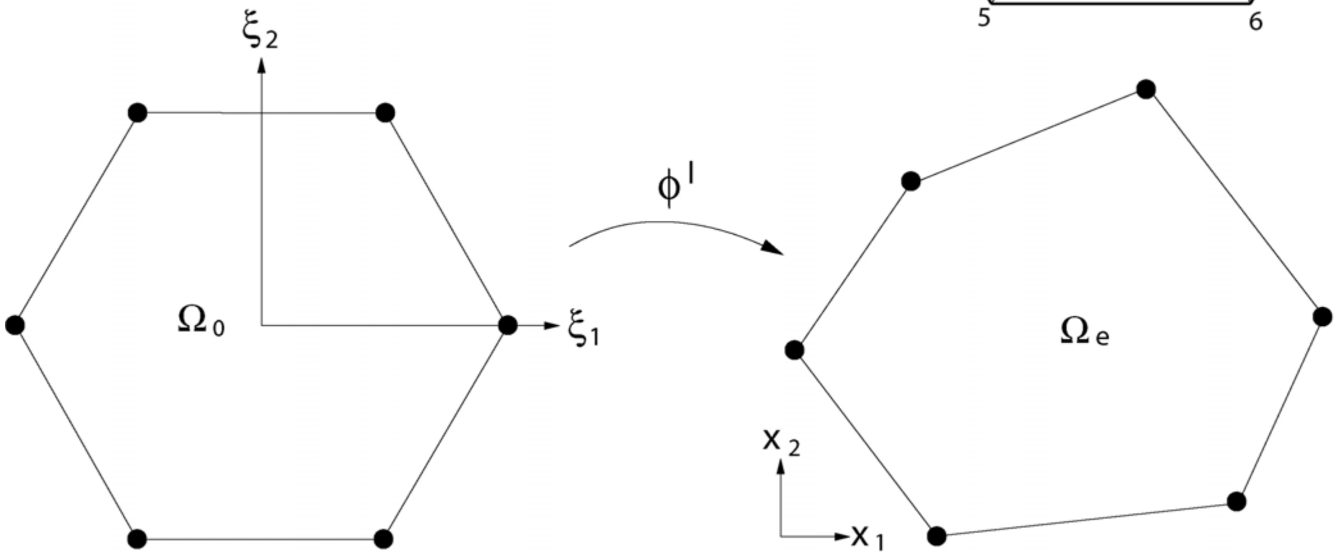
Canonical Elements



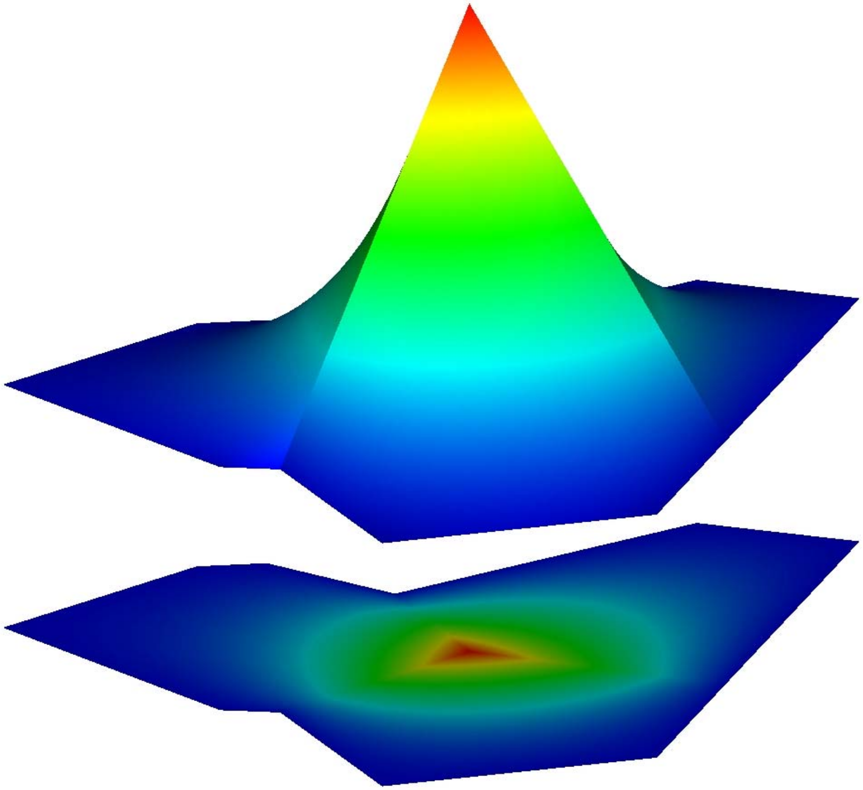
Polygon Interpolant Using Affine Mapping

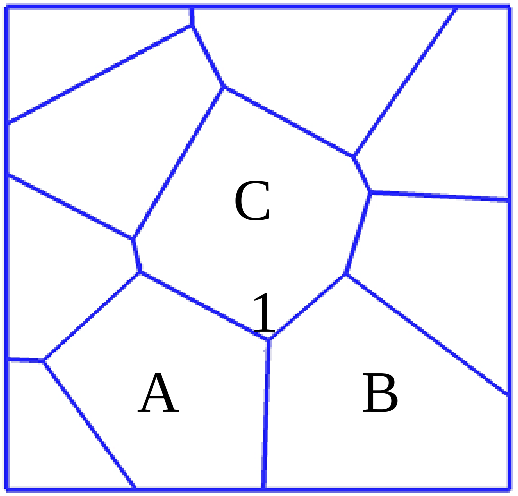
Laplace Shape Function





Polygonal (Laplace) Basis Function





Principle of Maximum Uncertainty/Entropy

(Shannon,1948; Jaynes,1957)

- Provides the least-biased solution when incomplete/ insufficient information is available
- For a discrete probability distribution $\{p_i\}, i = 1,2,\cdots,n,$ with $\sum p_i = 1$, let the average (expected) value of property E^r be known: $\sum_i p_i E_i^r = <E^r>$
- Maximizing the information-entropy $H(p_i) = -\sum_{i=1}^{n} p_i \log p_i$ subject to the constraints leads to the most probable solution (Gibbs-Boltzmann distribution in statistical mechanics)

Principle of Minimum Relative Entropy

(Kullback, 1959)

- Given a prior distribution $\mathbf{q}$, the Kullback-Leibler distance (mutual information) between $\mathbf{p}$ and $\mathbf{q}$ is

$$D(\mathbf{p} \,|\, \mathbf{q}) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}, \quad I(X;Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)}$$

- Minimizing the relative-entropy with a uniform prior, $q_i = 1/n$, is equivalent to maximizing Shannon entropy

- Other measures: Renyi and Tsallis entropies

MAXENT at Work!

- Coin toss: $p_1 + p_2 = 1$ and **MAXENT** gives $p_1 = p_2 = 1/2$
  $\equiv$ Principle of indifference or insufficient reason

- Wallis provided a combinatorial justification for the choice of the specific form of H(**p**)

- Suppose a die has been tossed N times and we are told that the average number of spots is **4.5** and not **3.5** (*honest die*). Then, **MAXENT** gives

$$\{p_1, p_2, \ldots, p_6\} = \{0.054,\ 0.079,\ 0.114,\ 0.165,\ 0.240,\ 0.347\}$$

MAXENT Applications

- Statistical mechanics and physics
- Communication and natural language modeling
- Image reconstruction and biology (protein folding)
- Economics and urban planning
- Materials science (crystallography/microstructure)
- . . . and many more where *uncertainty* resides

MAXENT in Computational Mechanics

- Elegant and least-biased solution for scattered data approximation by associating shape functions with discrete probability measures

- Broader implications in computational mechanics:
  - Numerical estimation/prediction
  - Tailored approximants for meshfree methods
  - Microstructural design and optimization
  - Ill-posed (non-unique) inverse problems
  - Multiscale modeling

Problem Statement: MAXENT Shape Functions

$$\operatorname*{Max}_{\boldsymbol{\varphi}} \quad H(\phi_i) = -\sum_{i=1}^{n} \phi_i \log \phi_i \quad \text{s.t.}$$

$$\left.\begin{aligned} &\sum_{i=1}^{n} \phi_i = 1 \\ &\sum_{i=1}^{n} \phi_i x_i = x \\ &\sum_{i=1}^{n} \phi_i y_i = y \end{aligned}\right\} \qquad \underset{(3\times n)}{\mathbf{P}} \; \underset{(n\times 1)}{\boldsymbol{\varphi}} = \underset{(3\times 1)}{\mathbf{p}}$$

Constraints

ϕ_i : `Probability of influence' of node *i* at **x**

Minimum Norm Solution

General Solution of $\mathbf{P}\boldsymbol{\varphi} = \mathbf{p}$:

$$\boldsymbol{\varphi} = \mathbf{P}^{+}\mathbf{p} + (\mathbf{I} - \mathbf{P}^{+}\mathbf{P})\mathbf{c}$$

and if **c = 0** we obtain the min-norm solution:

$$\boldsymbol{\varphi} = \mathbf{P}^{+}\mathbf{p}, \quad \mathbf{P}^{+} \equiv$$ Generalized Inverse

which is the solution of $\text{Min}\left(H(\boldsymbol{\varphi}) = \left\|\boldsymbol{\varphi}\right\|_2\right)$ $s.t.\ \mathbf{P}\boldsymbol{\varphi} = \mathbf{p}$

Since $\phi_i < 0$ is possible, $H(\bullet)$ is not suitable as an uncertainty measure

MLS and Weighted Minimum-Norm Solution

MLS: $\text{Min } \| \mathbf{W}^{1/2}(\mathbf{P}^T\mathbf{a}-\mathbf{u}) \|_2^2 \;\Rightarrow\; \mathbf{A}\mathbf{a}=\mathbf{B}\mathbf{u}$

$$\mathbf{A}=\mathbf{P}\mathbf{W}\mathbf{P}^{\mathrm{T}},\quad \mathbf{B}=\mathbf{P}\mathbf{W} \qquad \boxed{\boldsymbol{\varphi}=\mathbf{B}^T\mathbf{A}^{-1}\mathbf{p}=\mathbf{W}\mathbf{P}^T\boldsymbol{\gamma}}$$

dual variables

Primal Problem: $\text{Min}\left(\boldsymbol{\varphi}^T\mathbf{W}^{-1}\boldsymbol{\varphi}\right)\ s.t.\ \mathbf{P}\boldsymbol{\varphi}=\mathbf{p}$

Let $\boldsymbol{\psi}=\mathbf{W}^{-1/2}\boldsymbol{\varphi},\ \mathbf{Q}=\mathbf{P}\mathbf{W}^{1/2}$, then

$$\text{Min}\ \|\boldsymbol{\psi}\|_2^2\ \ s.t.\ \mathbf{Q}\boldsymbol{\psi}=\mathbf{p}$$

$$\boxed{\boldsymbol{\varphi}=\mathbf{W}^{1/2}\mathbf{Q}^{+}\mathbf{p}\ \ (\mathbf{Q}^{+}\text{: } \textit{Matlab function } \textbf{pinv})}$$

MAXENT Solution Using Lagrange Multipliers

- First variation of augmented Lagrangian is zero $(\delta L = 0)$

$$L = -\sum_{i=1}^{n} \phi_i \log \phi_i + \lambda_0 \left(1 - \sum_i \phi_i \right) + \lambda_1 \left(x - \sum_i \phi_i x_i \right)$$

$$+ \lambda_2 \left(y - \sum_i \phi_i y_i \right)$$

$$\delta L = (-1 - \log \phi_i - \lambda_0 - \lambda_1 x_i - \lambda_2 y_i)\delta\phi_i = 0 \quad \forall\, \delta\phi_i$$

and since the variations $\delta\phi_i$ are arbitrary

$$-1 - \log \phi_i - \lambda_0 - \lambda_1 x_i - \lambda_2 y_i = 0 \quad (i = 1, 2, \ldots, n)$$

MAXENT Solution (Cont'd)

- Letting $\lambda_0 = \log Z - 1$ (Z is the partition function), we get

$$\log \phi_i + \log Z = -\lambda_1 x_i - \lambda_2 y_i$$

- Since $\sum_i \phi_i = 1$,

$$\phi_i = \frac{e^{-\lambda_1 x_i - \lambda_2 y_i}}{Z}, Z = \sum_{j=1}^{n} e^{-\lambda_1 x_j - \lambda_2 y_j}$$

MAXENT Solution (Cont'd)

- If only one constraint exists ($\lambda_1 = \lambda_2 = 0$), then $Z = n$

$$\phi_i = \frac{1}{n} \forall i$$ (nearest-neighbor interpolant)

- In general, λ_1 and λ_2 satisfy two non-linear equations:

$$-\frac{\partial(\log Z)}{\partial \lambda_1} = x \quad \Leftrightarrow \quad \frac{\sum_{i=1}^{n} e^{-\lambda_1 x_i - \lambda_2 y_i} x_i}{Z} - x = 0$$

$$-\frac{\partial(\log Z)}{\partial \lambda_2} = y \quad \Leftrightarrow \quad \frac{\sum_{i=1}^{n} e^{-\lambda_1 x_i - \lambda_2 y_i} y_i}{Z} - y = 0$$

Numerical Algorithm for MAXENT Shape Functions

- Let $\tilde{x}_i = x_i - x, \; \tilde{y}_i = y_i - y$. Then,

$$f_1(\lambda_1, \lambda_2) = \frac{\partial(\log \tilde{Z})}{\partial \lambda_1} = 0 \quad \Leftrightarrow \quad -\frac{\sum_{i=1}^{n} e^{-\lambda_1 \tilde{x}_i - \lambda_2 \tilde{y}_i} \tilde{x}_i}{\tilde{Z}} = 0$$

$$f_2(\lambda_1, \lambda_2) = \frac{\partial(\log \tilde{Z})}{\partial \lambda_2} = 0 \quad \Leftrightarrow \quad -\frac{\sum_{i=1}^{n} e^{-\lambda_1 \tilde{x}_i - \lambda_2 \tilde{y}_i} \tilde{y}_i}{\tilde{Z}} = 0$$

- The vector field **f** is the gradient of a scalar potential:

$$F = \log \tilde{Z}(\lambda_1, \lambda_2), \quad \mathbf{f} = \nabla F$$

Numerical Algorithm (Cont'd)

- Recast the MAXENT formulation as a convex minimizer (dual) problem **(Agmon et al., JCP, 1979):**

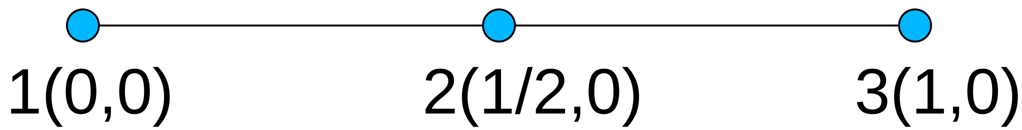
$$\text{Find } (\lambda_1, \lambda_2) \text{ s.t. } F = \log \tilde{Z}(\lambda_1, \lambda_2) \text{ is minimized}$$

- Initial guess $\boldsymbol{\lambda}^0 = \mathbf{0}$ Update $\lambda_r^{k+1} = \lambda_r^k + \alpha \Delta \lambda_r^k, \quad \Delta \boldsymbol{\lambda}^k = -\nabla F$

- α is determined by the condition that $F(\lambda_1^{k+1}, \lambda_2^{k+1})$ is minimized along the search direction

- Convergence criterion: $\left\| \nabla F \right\|^{\{k\}} < 10^{-7}$

MAXENT Shape Functions in 1D



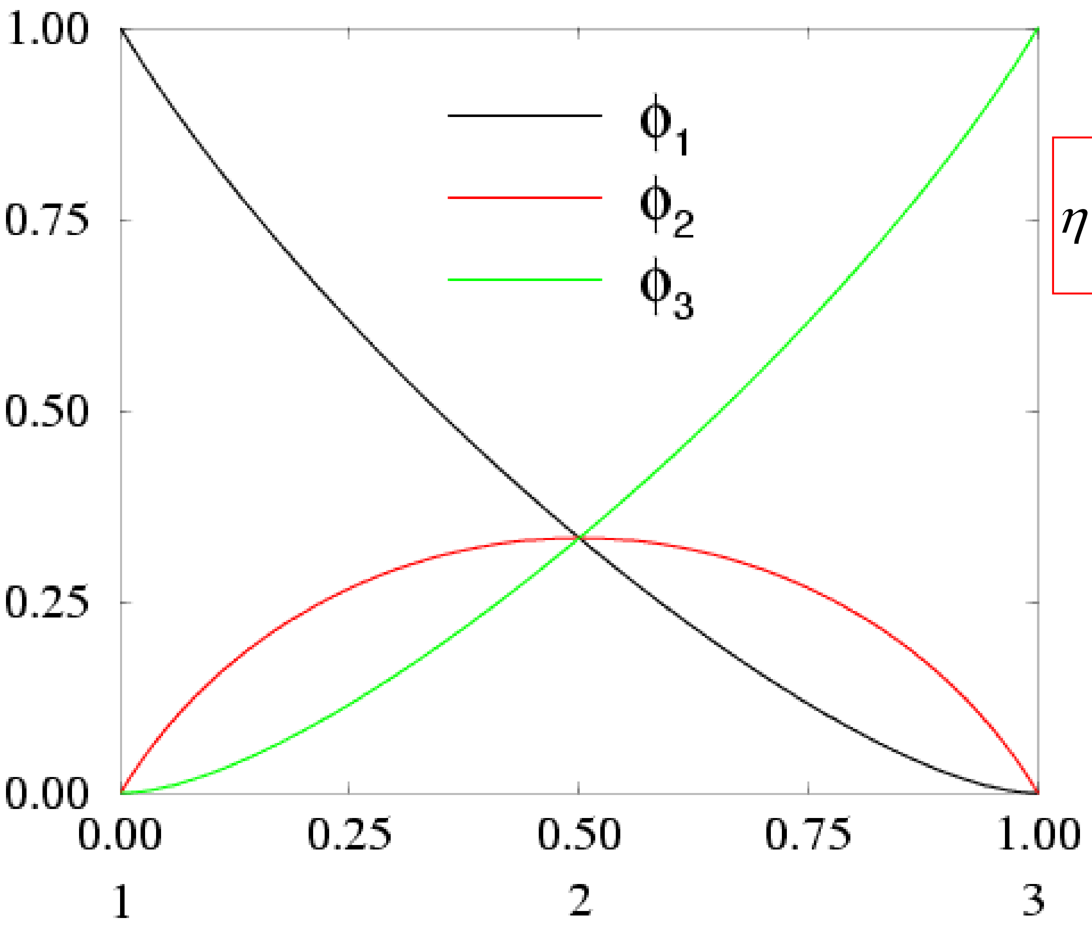
$$\phi_i = \frac{e^{-\lambda_1 x_i}}{Z}, \; Z = \sum_{j=1}^{3} e^{-\lambda_1 x_j}$$

$$\boxed{x_1 = 0, \; x_2 = 1/2, \; x_3 = 1}$$

$$Z = 1 + e^{-\lambda_1/2} + e^{-\lambda_1}, \;\; \phi_1 = \frac{1}{Z}, \; \phi_2 = \frac{e^{-\lambda_1/2}}{Z}, \; \phi_3 = \frac{e^{-\lambda_1}}{Z}$$

$$\sum_{i=1}^{3} \phi_i x_i = x : \;\; \boxed{\frac{\eta}{2} + \eta^2 = x(1 + \eta + \eta^2), \;\; \eta = e^{-\lambda_1/2}}$$

MAXENT Shape Functions in 1D (Cont'd)



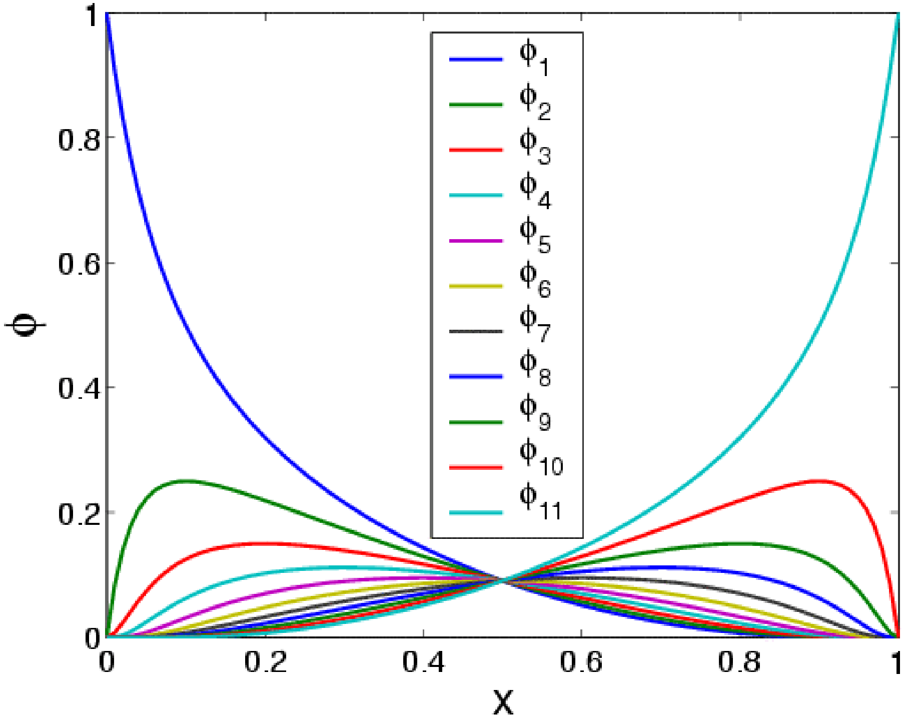
$$\eta = \frac{2x - 1 + \sqrt{12x(1-x)+1}}{4(1-x)}$$

$$\phi_1 = \frac{1}{1+\eta+\eta^2}$$

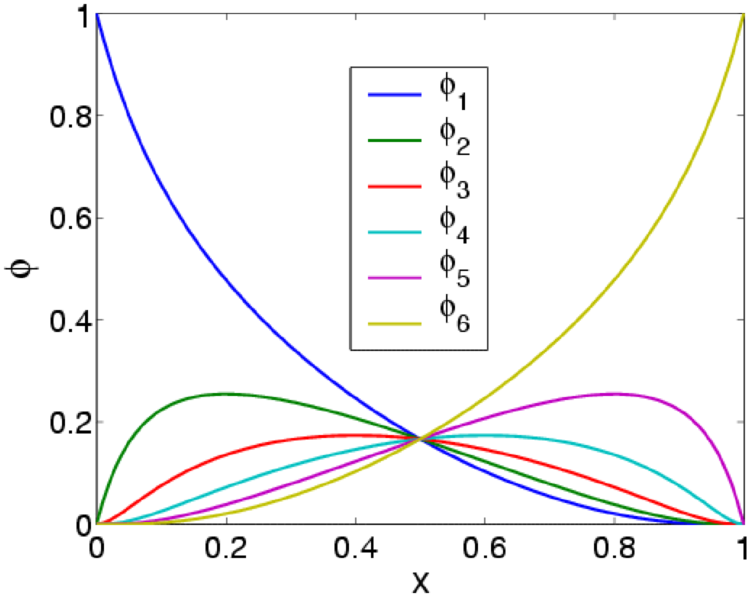
$$\phi_2 = \frac{\eta}{1+\eta+\eta^2}$$

$$\phi_3 = \frac{\eta^2}{1+\eta+\eta^2}$$

MAXENT Shape Functions in 1D (Cont'd)

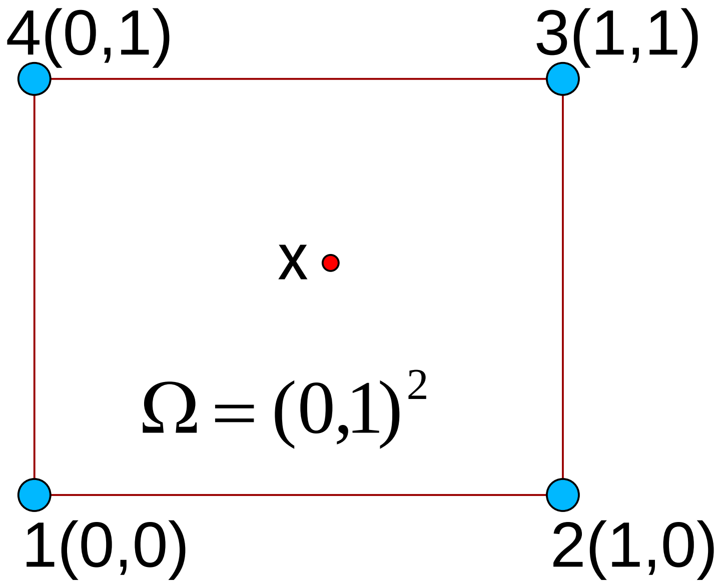


n = 11



n = 6

Square: MAXENT Shape Functions



$$Z = \sum_{j=1}^{4} e^{-\lambda_1 x_j - \lambda_2 y_j}$$

$$= 1 + e^{-\lambda_1} + e^{-\lambda_2} + e^{-\lambda_1 - \lambda_2}$$

$$\frac{e^{-\lambda_1} + e^{-\lambda_1 - \lambda_2}}{Z} = x$$

$$\frac{e^{-\lambda_2} + e^{-\lambda_1 - \lambda_2}}{Z} = y$$

which simplifies to

$$\frac{e^{-\lambda_1}}{1 + e^{-\lambda_1}} = x, \quad \frac{e^{-\lambda_2}}{1 + e^{-\lambda_2}} = y \Rightarrow e^{-\lambda_1} = \frac{x}{1-x}, \quad e^{-\lambda_2} = \frac{y}{1-y}$$

Square (Cont'd)

Since $\phi_i = \dfrac{e^{-\lambda_1 x_i - \lambda_2 y_i}}{Z}, \;\; Z = \sum_{j=1}^{n} e^{-\lambda_1 x_j - \lambda_2 y_j},$

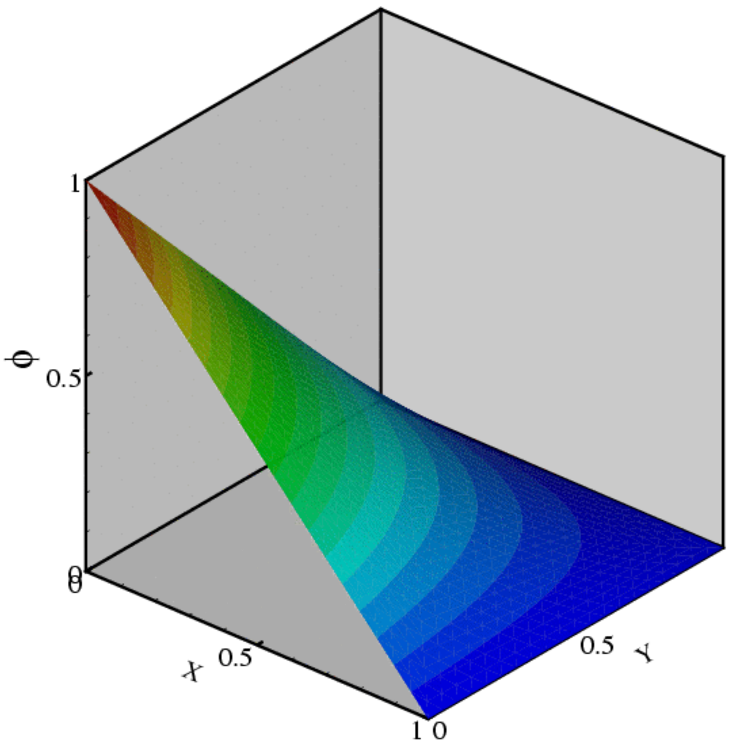
we obtain $Z^{-1} = (1-x)(1-y)$ and therefore

$$\phi_1(x,y) = (1-x)(1-y), \;\; \phi_2(x,y) = x(1-y),$$
$$\phi_3(x,y) = xy, \;\; \phi_4(x,y) = y(1-x)$$

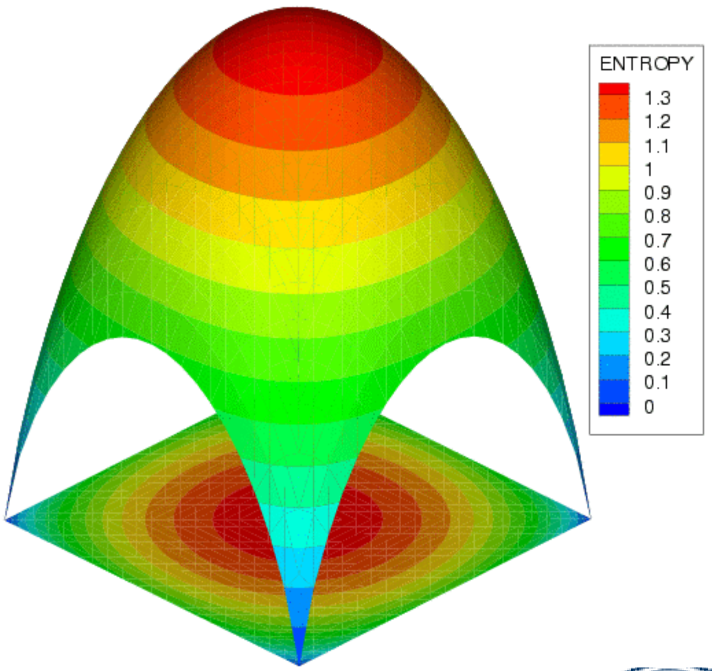
which are the same as bilinear finite element shape functions

Square (Cont'd)

MAXENT ≡ Bilinear FE Interpolation

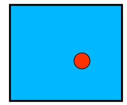


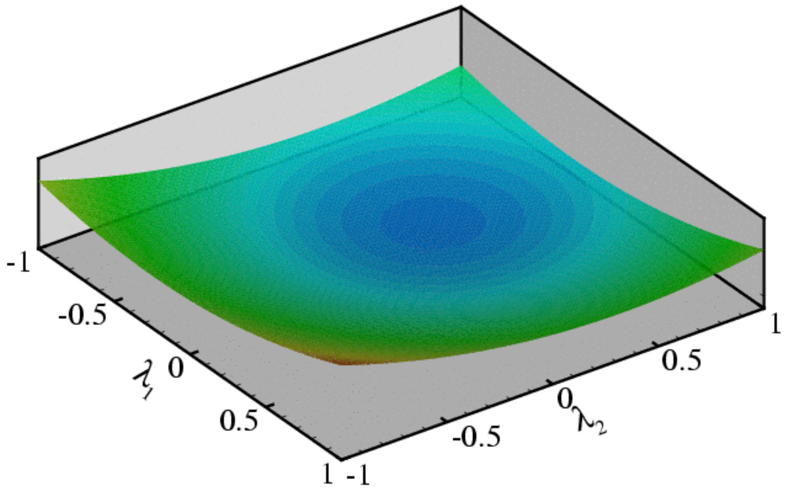
Shape Function

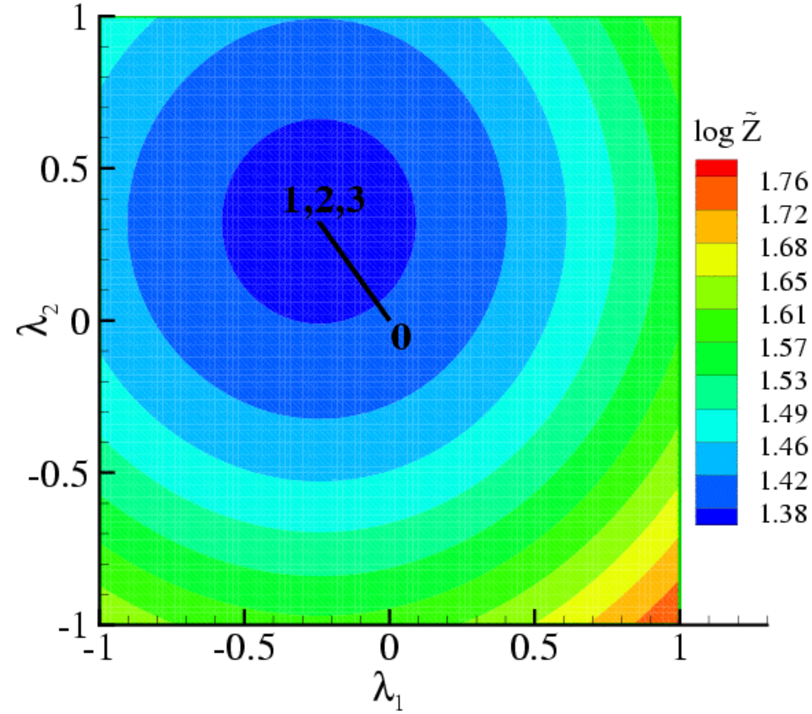


Entropy = log $\tilde{Z}$

Square: Convergence

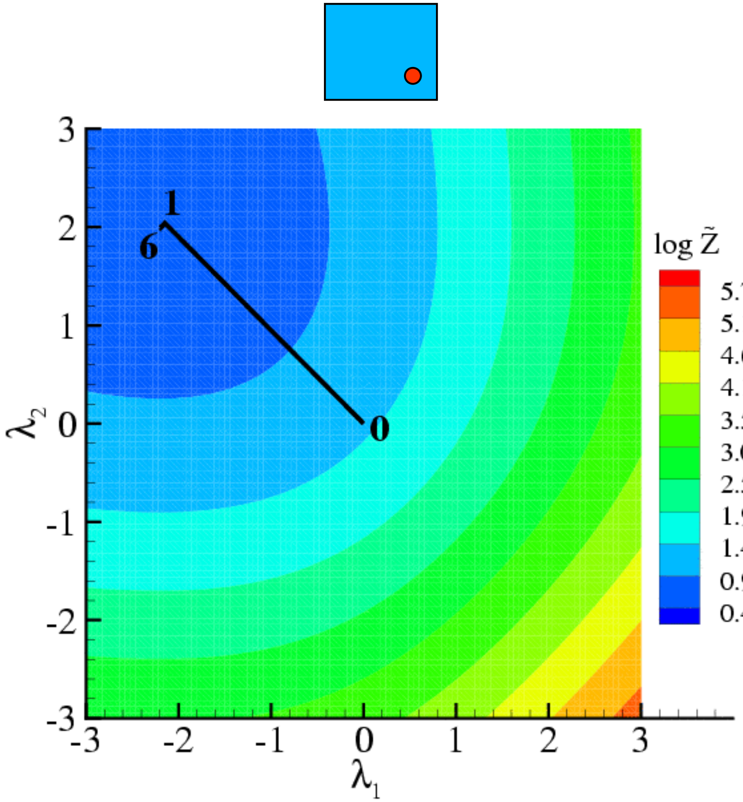


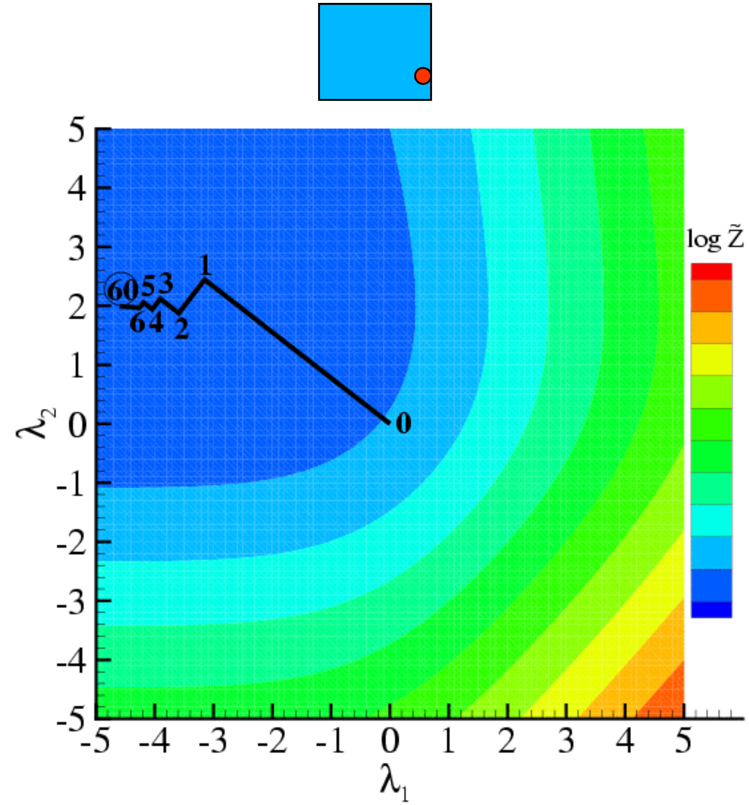




$$F = \log \tilde{Z} \text{ @ } x = (0.56, 0.42)$$

Square: Convergence (Cont'd)

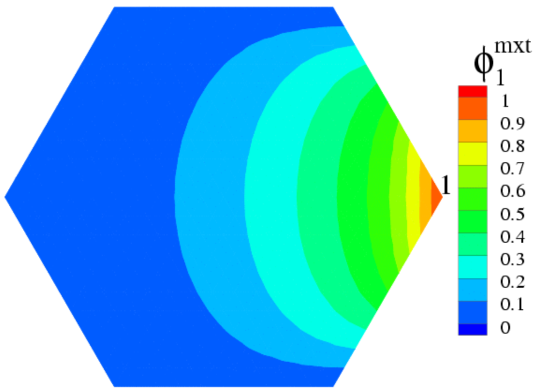




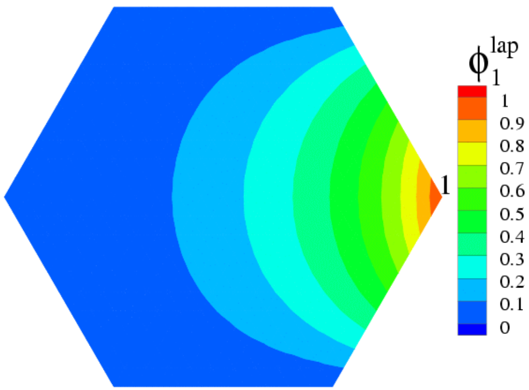
$F = \log \tilde{Z}$ @ $x = (0.9, 0.12)$ $F = \log \tilde{Z}$ @ $x = (0.99, 0.12)$

Use of nonlinear CG leads to faster convergence

Hexagon: Shape Functions

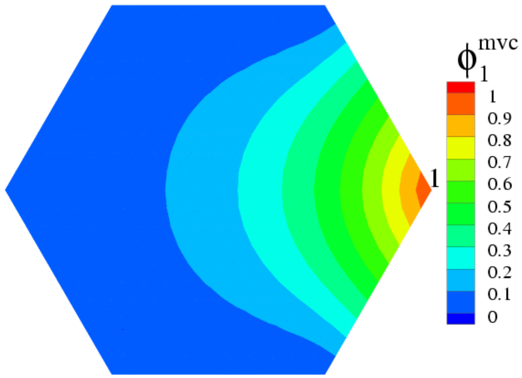


MAXENT

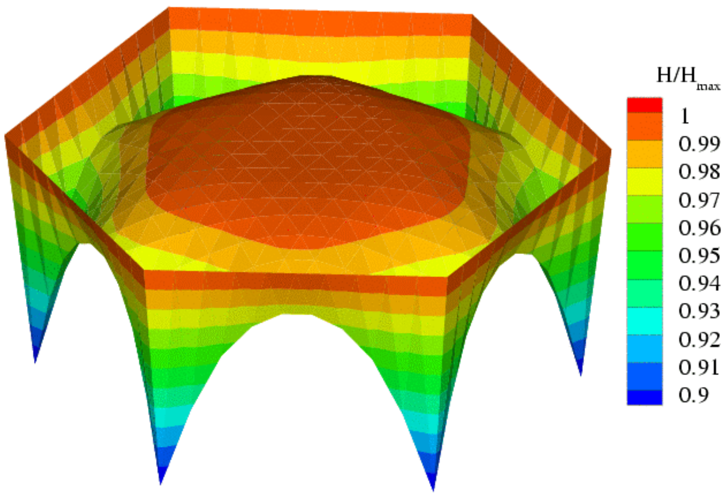


Laplace

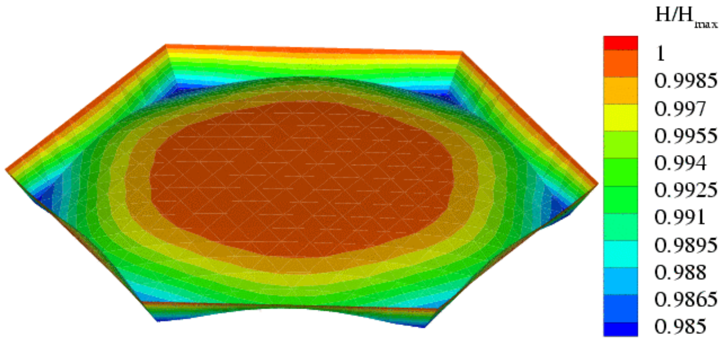
Mean-Value
Coordinates



Hexagon: Normalized Entropy

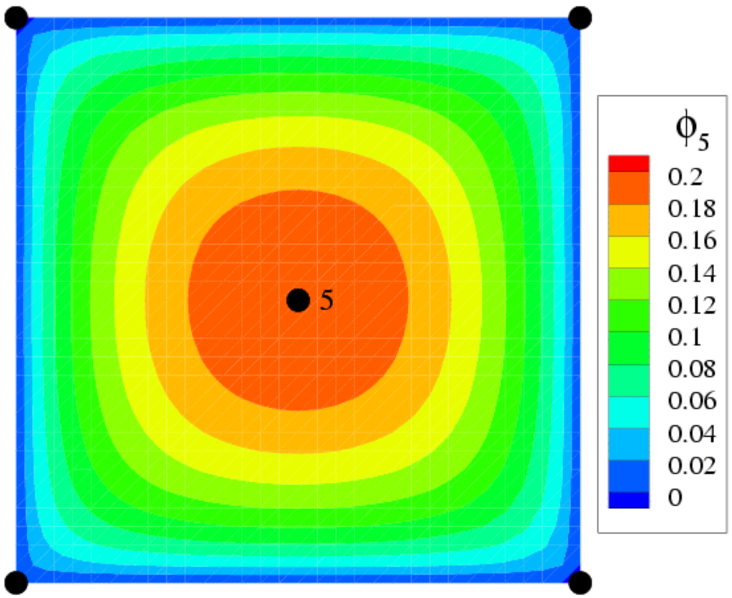


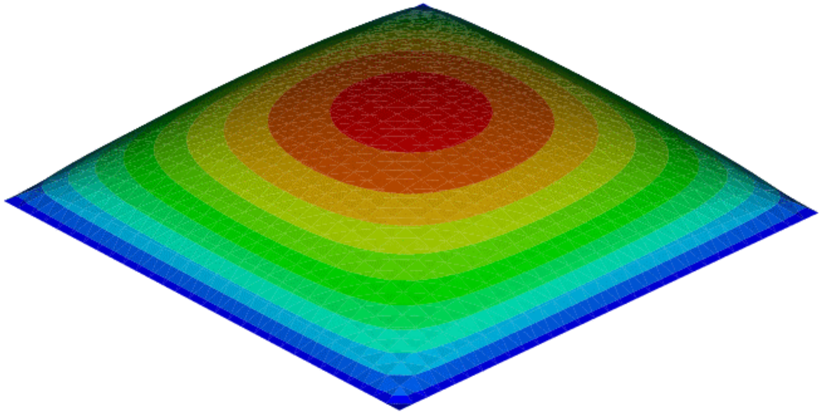
Mean-value coordinates



Laplace

Bubble (Shape) Function

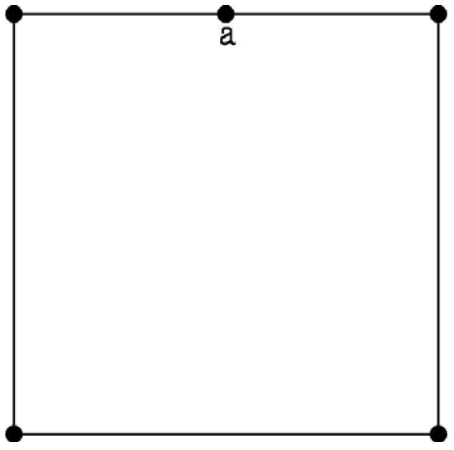




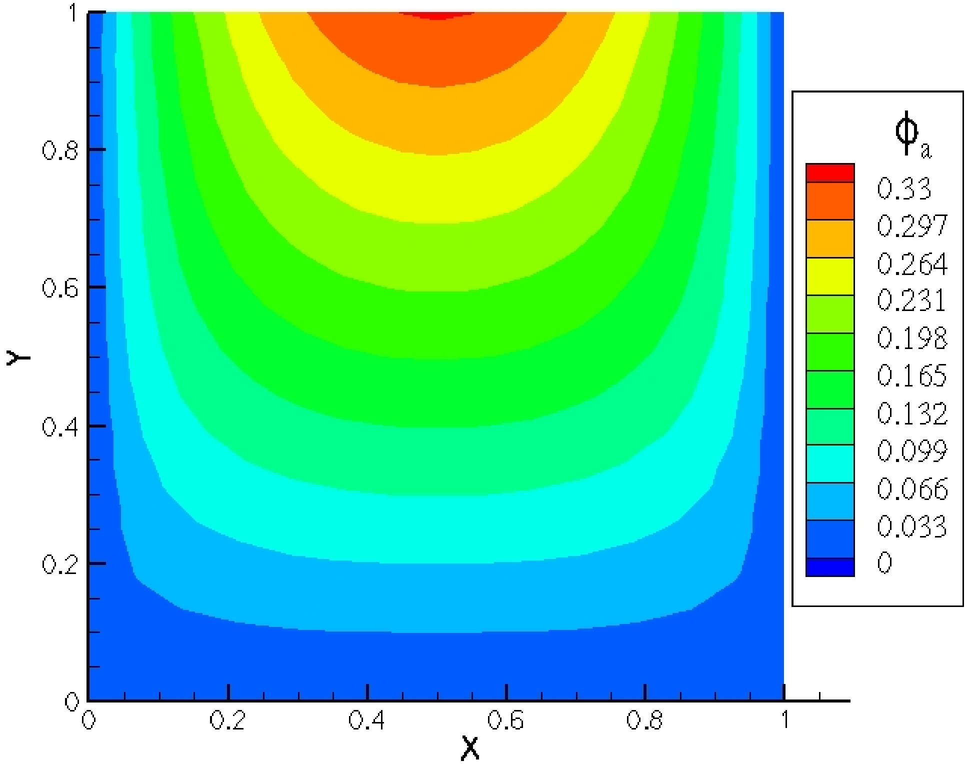
Contour plot

3D

Mid-Side Node: Shape Function

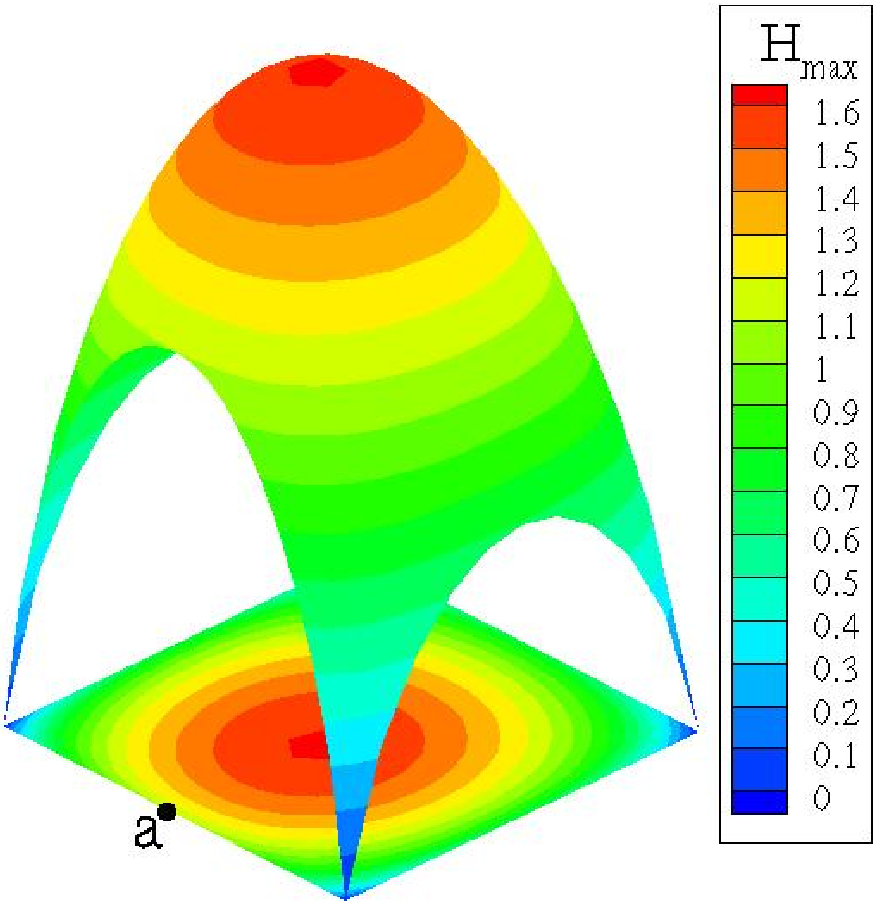


Five-node element



Shape function of node a

Mid-Side Node: Maximum Entropy Distribution



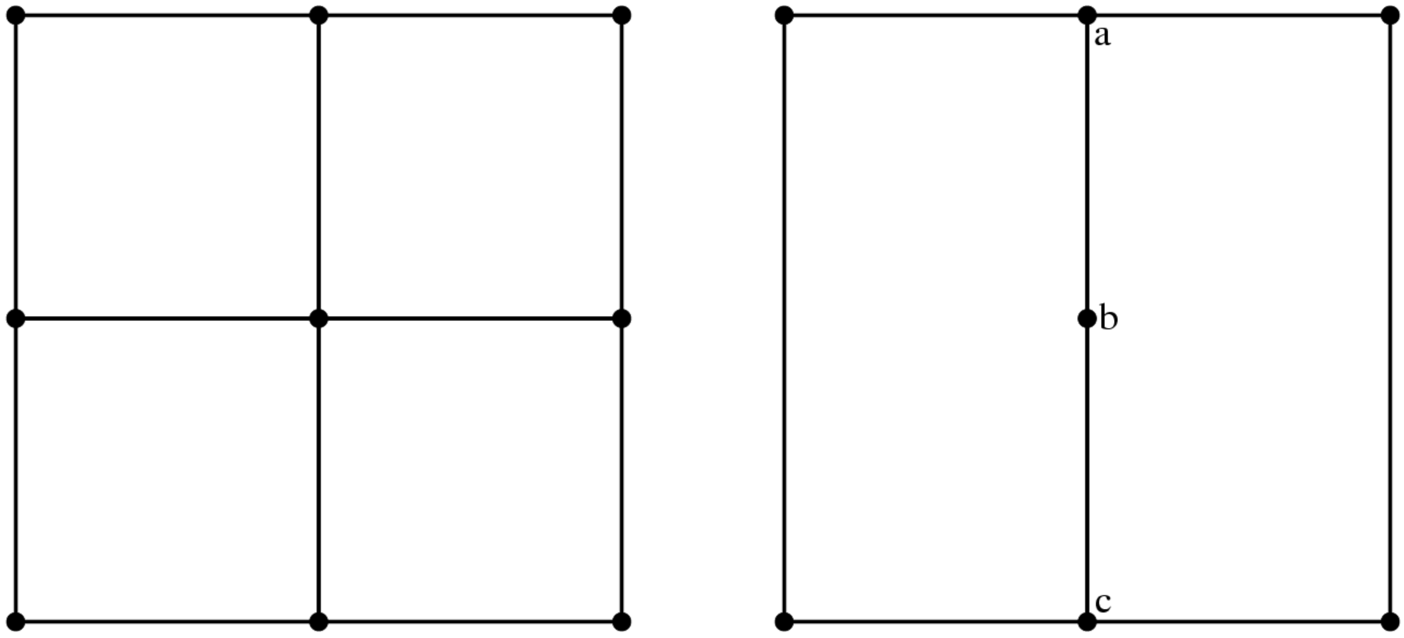
Shape Function (MAXENT) Derivatives

$$\frac{\partial \phi_i}{\partial \alpha} = \phi_i \left((x - x_i)\frac{\partial \lambda_1}{\partial \alpha} + (y - y_i)\frac{\partial \lambda_2}{\partial \alpha} \right), \quad \alpha = x, y$$

$$\begin{bmatrix} \dfrac{\partial \lambda_1}{\partial x} & \dfrac{\partial \lambda_1}{\partial y} \\ \dfrac{\partial \lambda_2}{\partial x} & \dfrac{\partial \lambda_2}{\partial y} \end{bmatrix} = - \begin{bmatrix} <x^2> - x^2 & <xy> - xy \\ <xy> - xy & <y^2> - y^2 \end{bmatrix}^{-1},$$

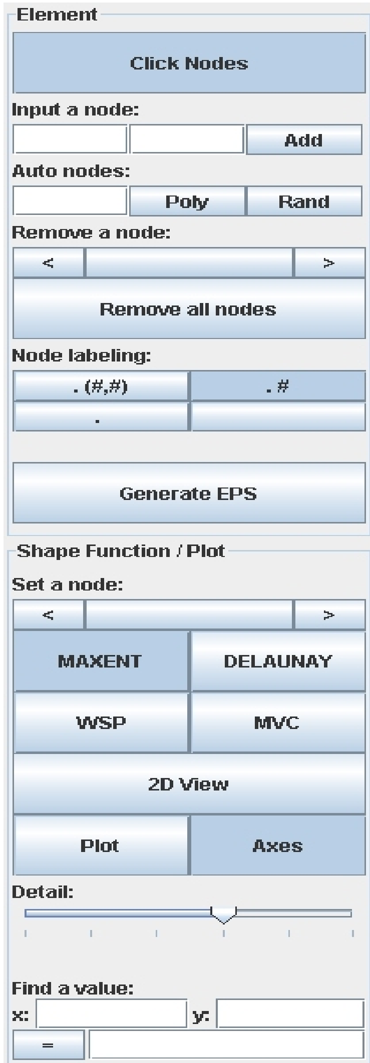
where $$<f> = \sum_{i=1}^{n} \phi_i f_i$$

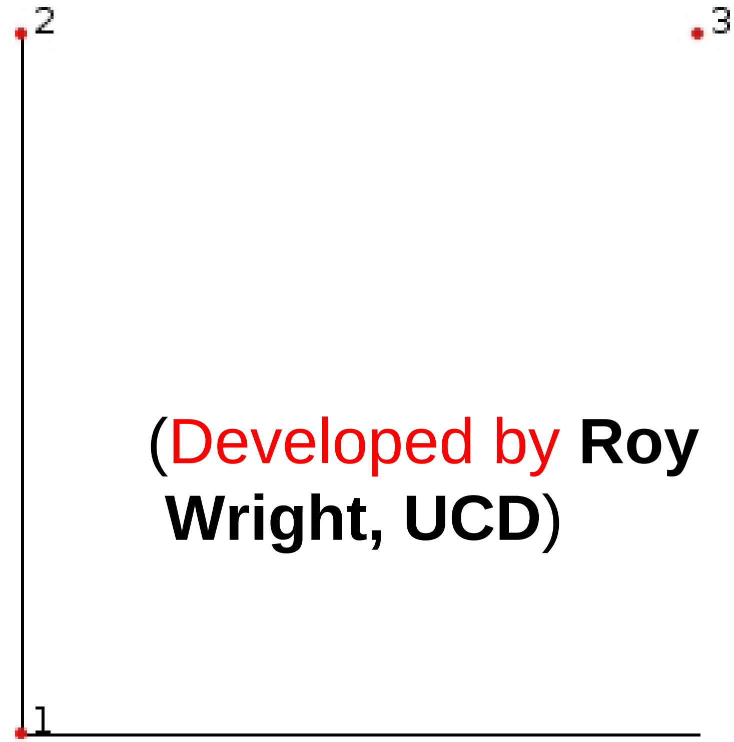
Galerkin Method (Patch Test)



Error norms: $$\frac{\|u-u^h\|_2}{\|u\|_2} \approx 10^{-8}, \quad \frac{\|u-u^h\|_E}{\|u\|_E} \approx 10^{-7}$$

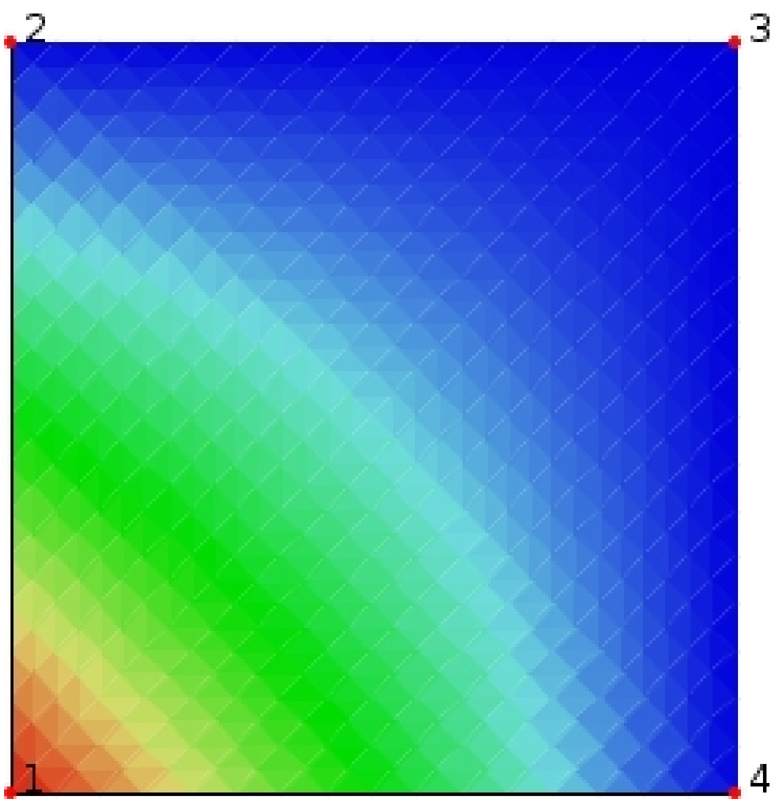
Shape Function Visualization: JAVA Applet

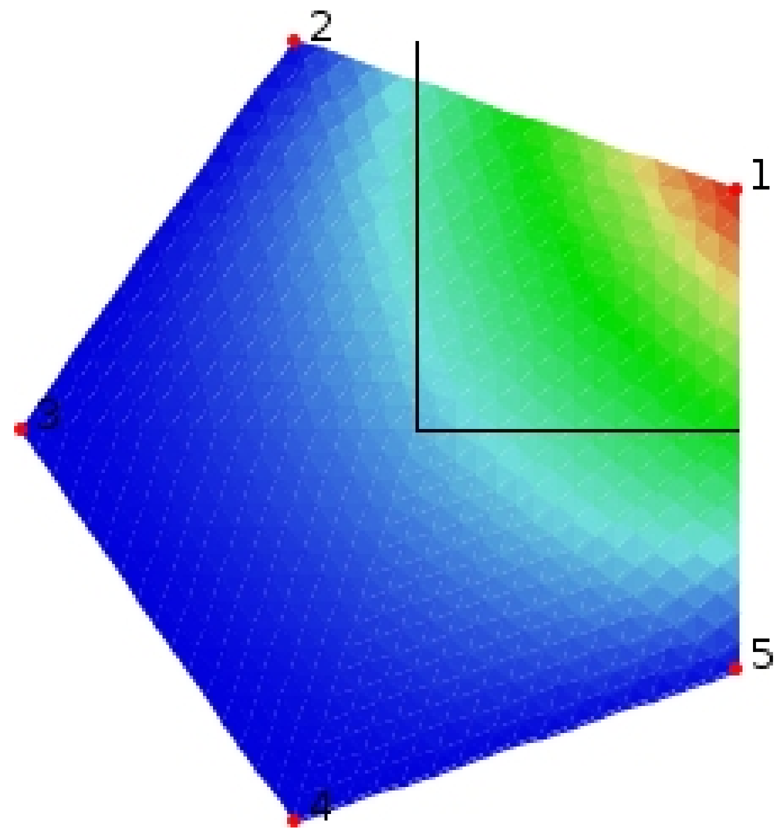




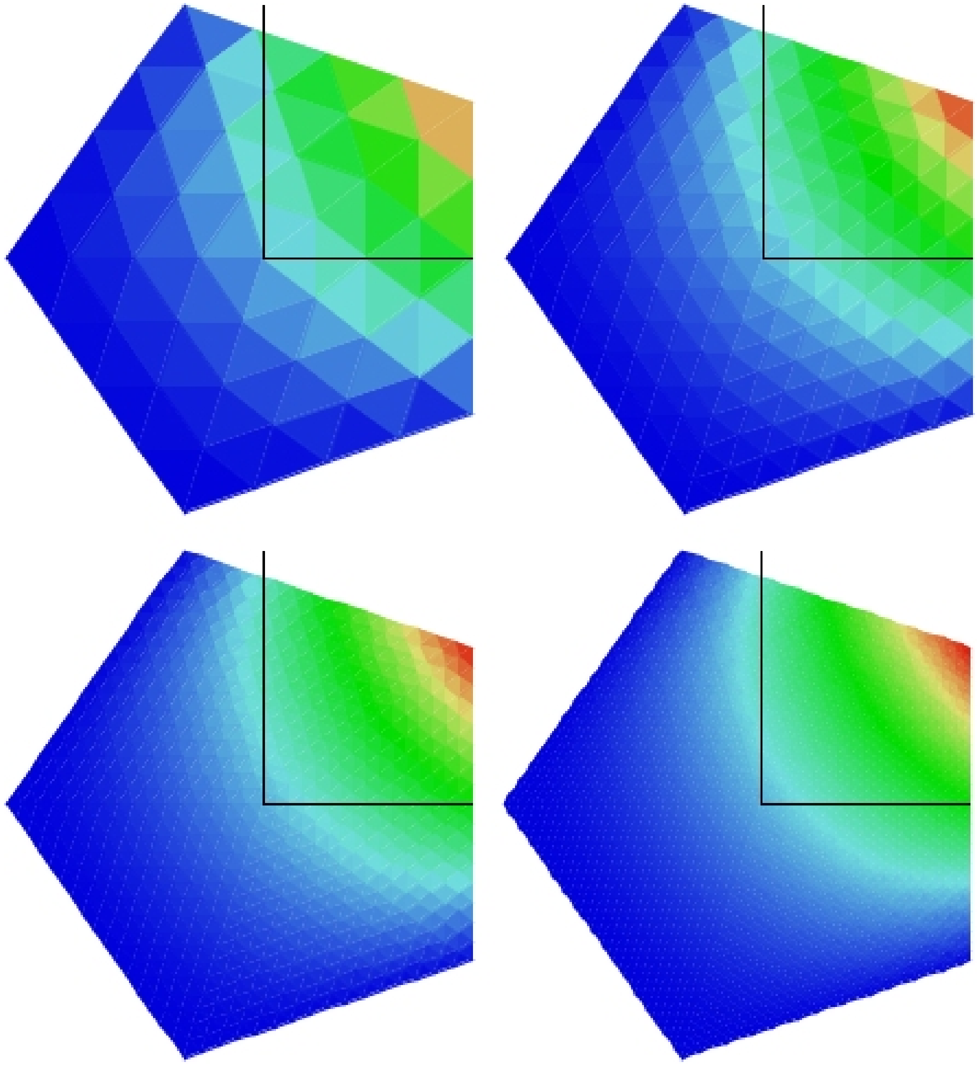
(Developed by **Roy Wright, UCD**)

JAVA Applet (Cont'd)

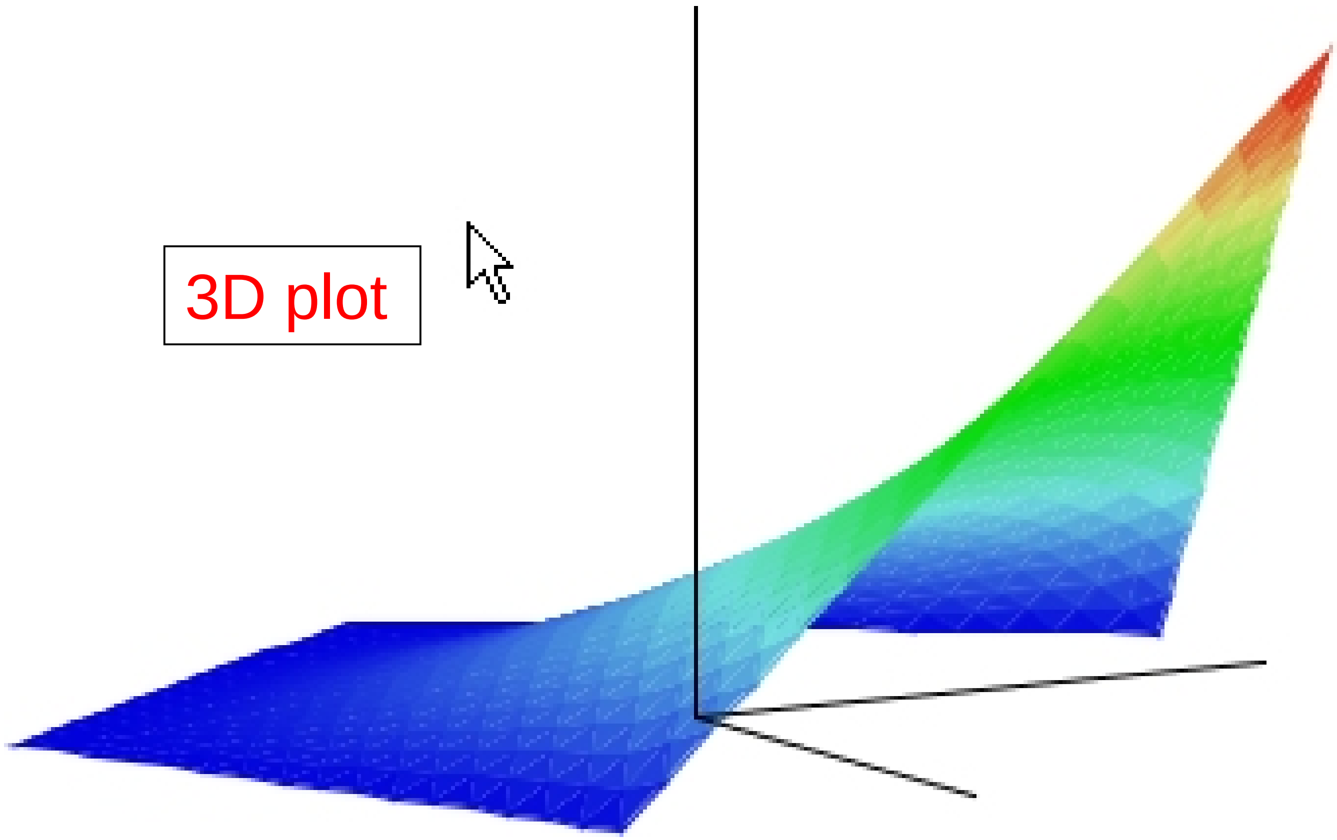




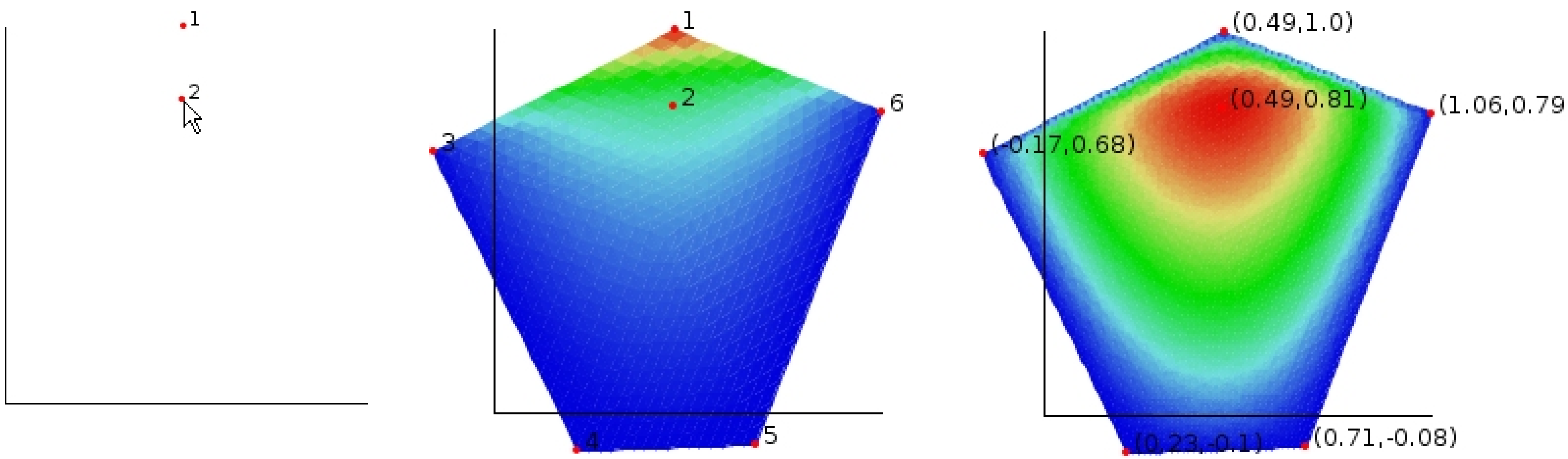
JAVA Applet (Cont'd)



JAVA Applet (Cont'd)

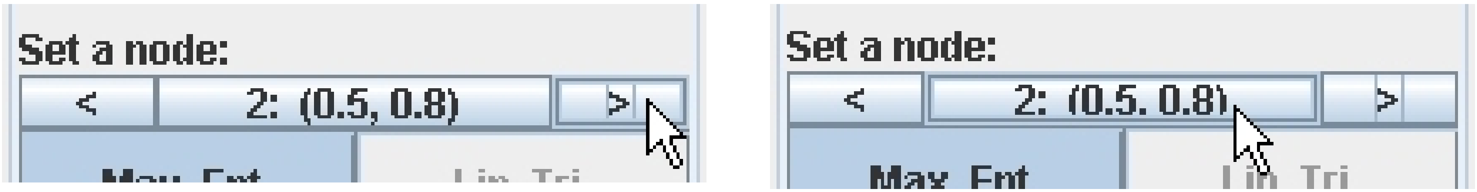


JAVA Applet (Cont'd)

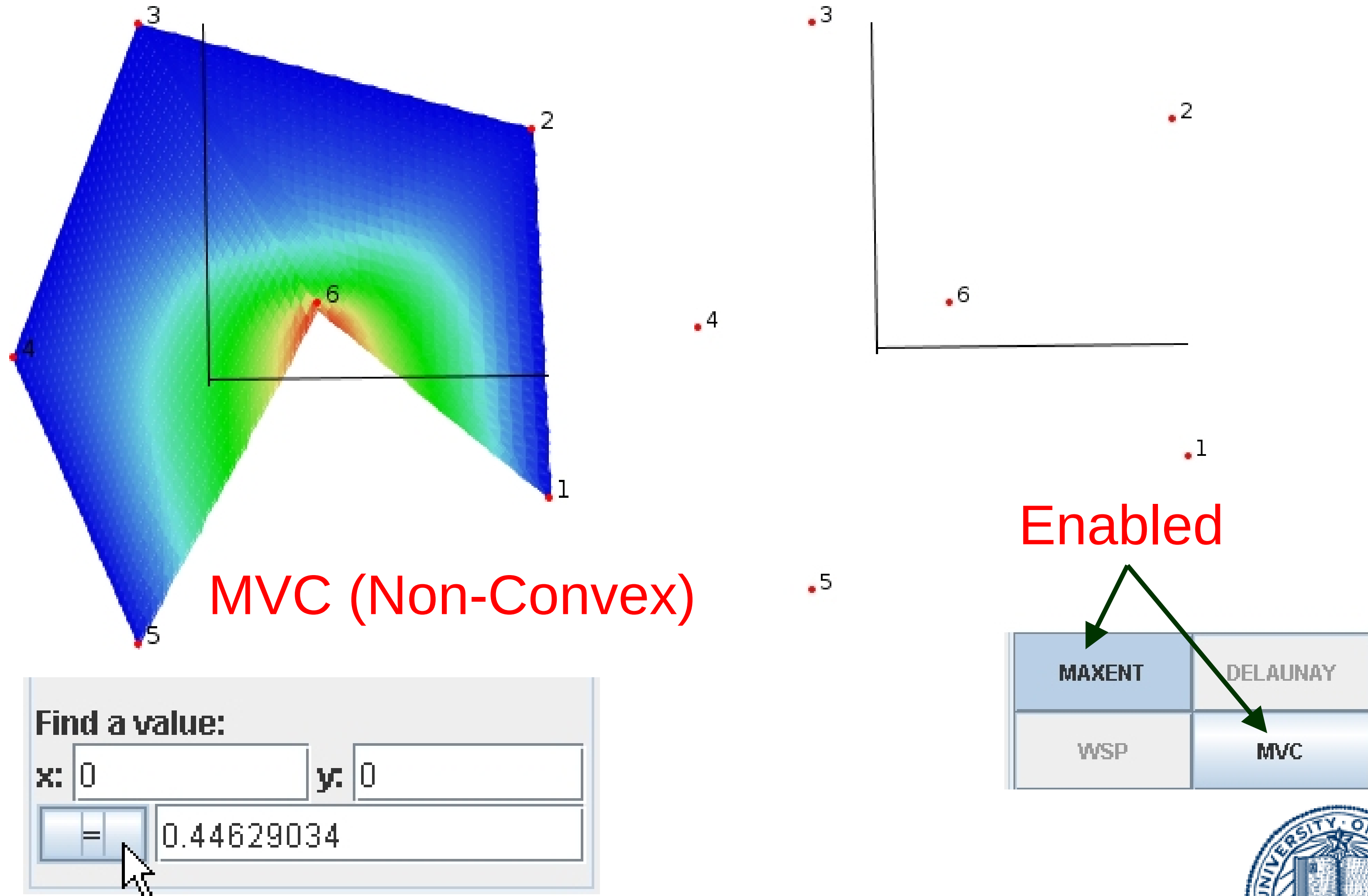


Mouse-click to insert

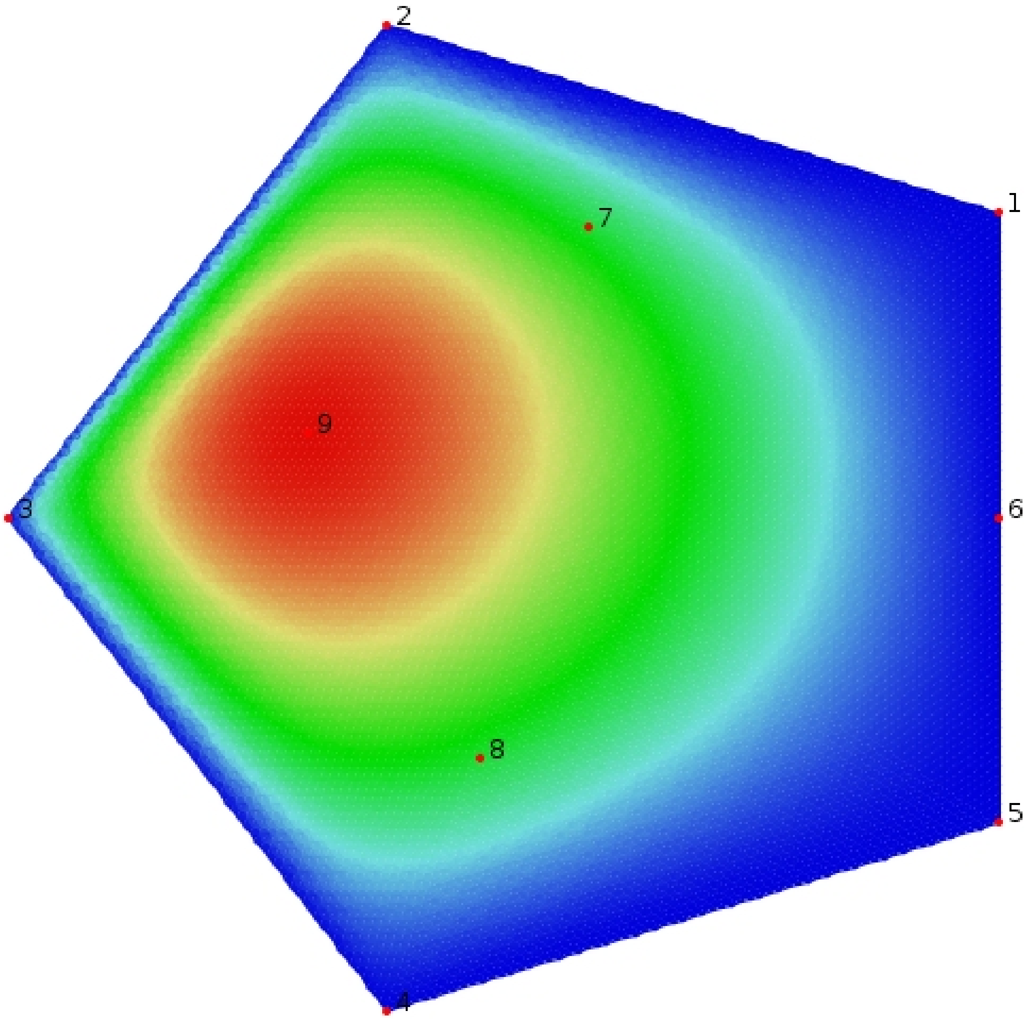
Right-click to delete



JAVA Applet (Cont'd)

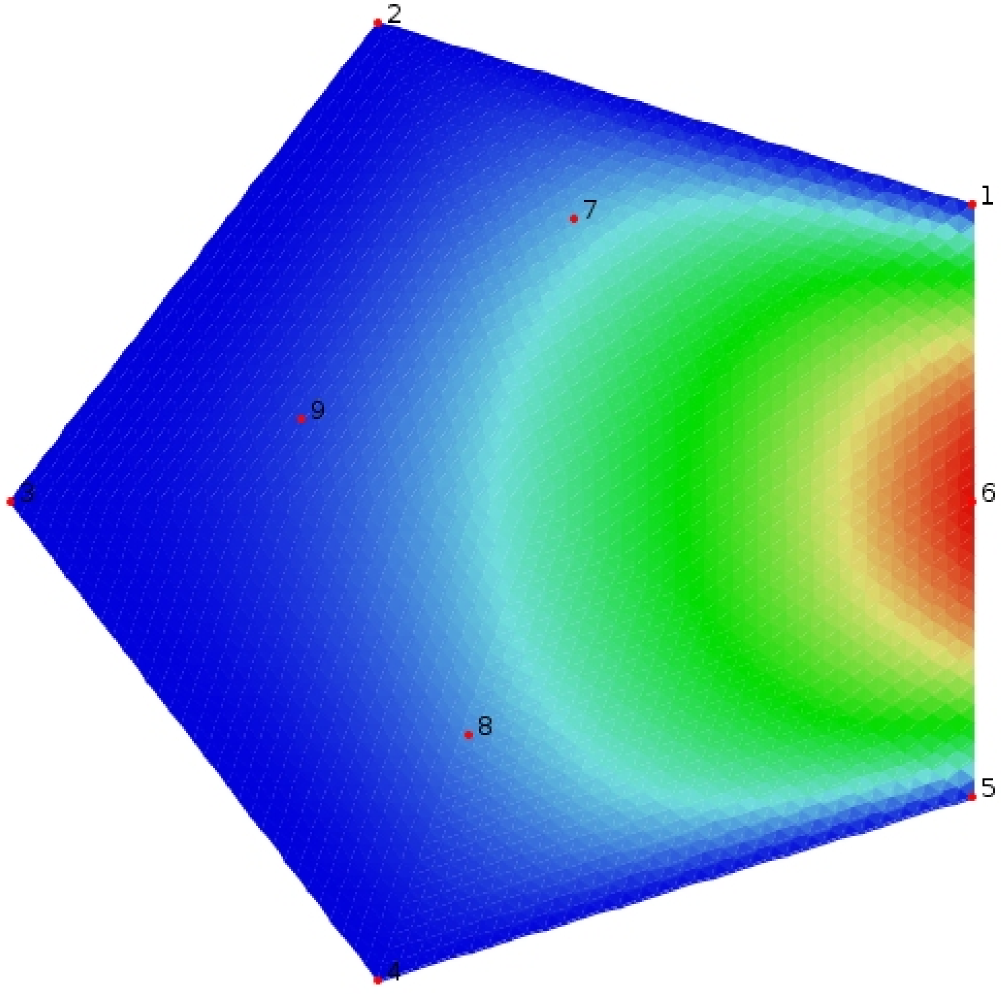


Visualization of Shape Functions



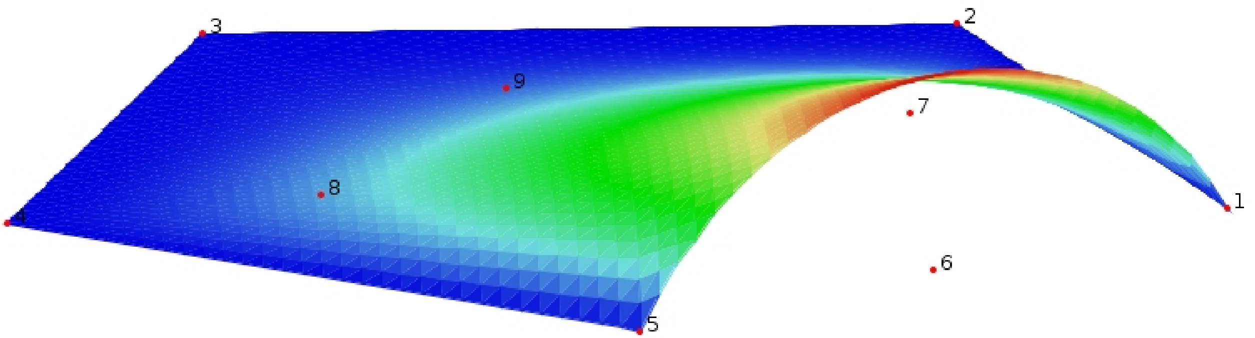
Contour plot

Side Node



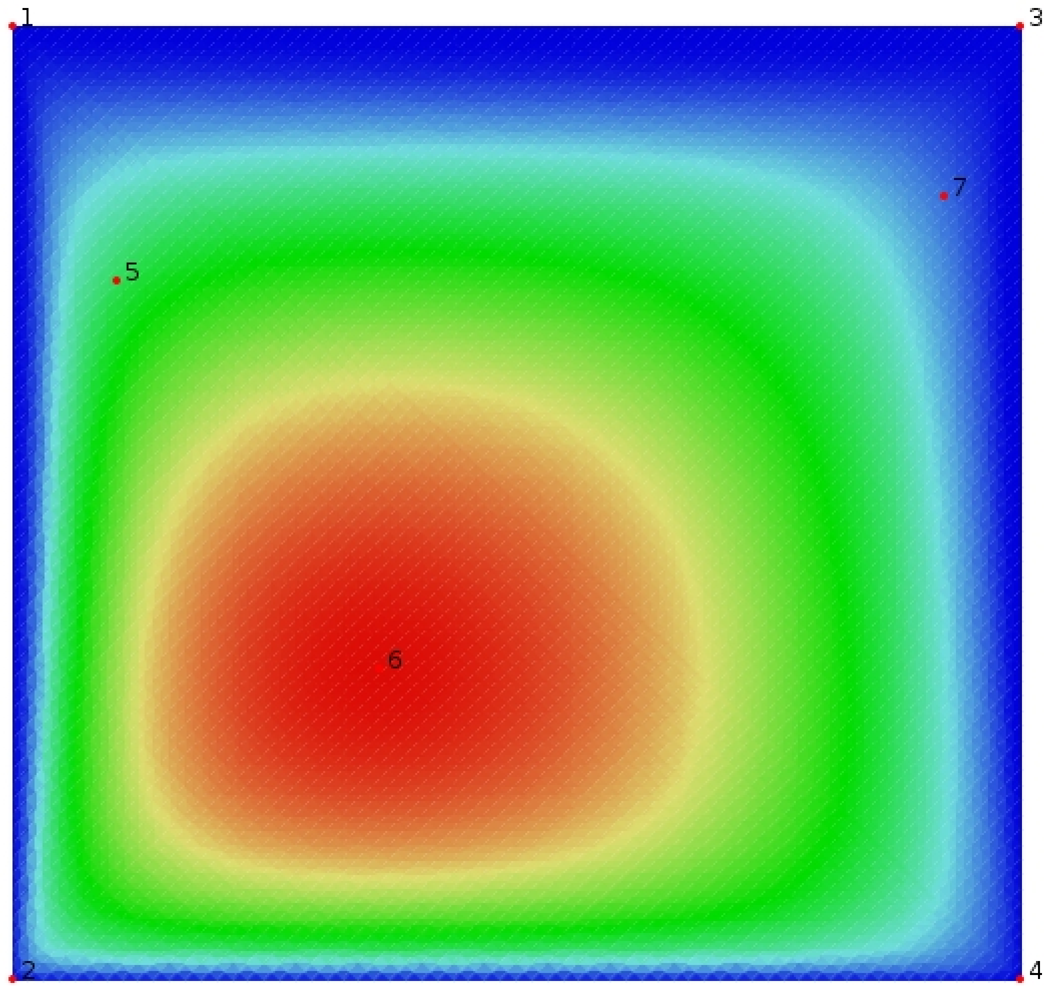
Contour plot

Side Node (Cont'd)

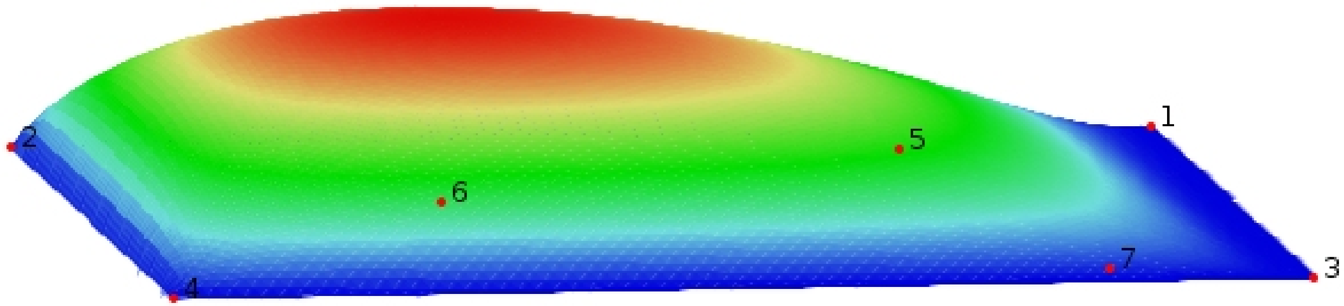


3D Plot

Interior Node



Interior Node (Cont'd)



3D Plot

Related Applications: A. Supervised Learning

(Gupta, Ph.D. thesis, Stanford, 2003)

Objective: Estimation of unknown quantities based on observed samples (numerical estimation), for e.g., pollutants in a city, spam e-mail, speech recognition

Feature Random Vector $X \in \mathbf{R}^d$ ⟷ relationship ⟷ Observation /Classification RV $Y \in \mathbf{R}$

For $d \uparrow$*, the curse of dimensionality!*

Given $P_{X,Y}$ and iid data $\{(X_1, Y_1), (X_2, Y_2), \ldots, (X_2, Y_2)\}$

ESTIMATE $P_{Y|X}$

A. Supervised Learning (LIME Algorithm)

Distortion function $D : \mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{R}^+$ is the mean squared error

$$D(r,s) = \frac{1}{d}\sum_{k=1}^{d}(r_k - s_k)^2$$

Compute weights $w_i(X)$ (partition of unity) by solving

$$\text{Minimize}\left[\text{D}\left(\sum_{\text{i}=1}^{\text{k}} \text{w}_{\text{j}} X_j, \text{X} \right) \text{-} \alpha \text{H}(\mathbf{w}) \right],$$

where α is chosen and k training samples are picked

$$\hat{P}_{Y|X}(g \mid x) = \sum_{i=1}^{k} w_i(x) I_{Y_i(X)=g}$$

B. Local MAXENT Meshfree Method

(Arroyo and Ortiz, 2005)

$$\text{Minimize } \left[\beta\, \mathrm{U}(\boldsymbol{\varphi}) - H(\boldsymbol{\varphi})\right]$$

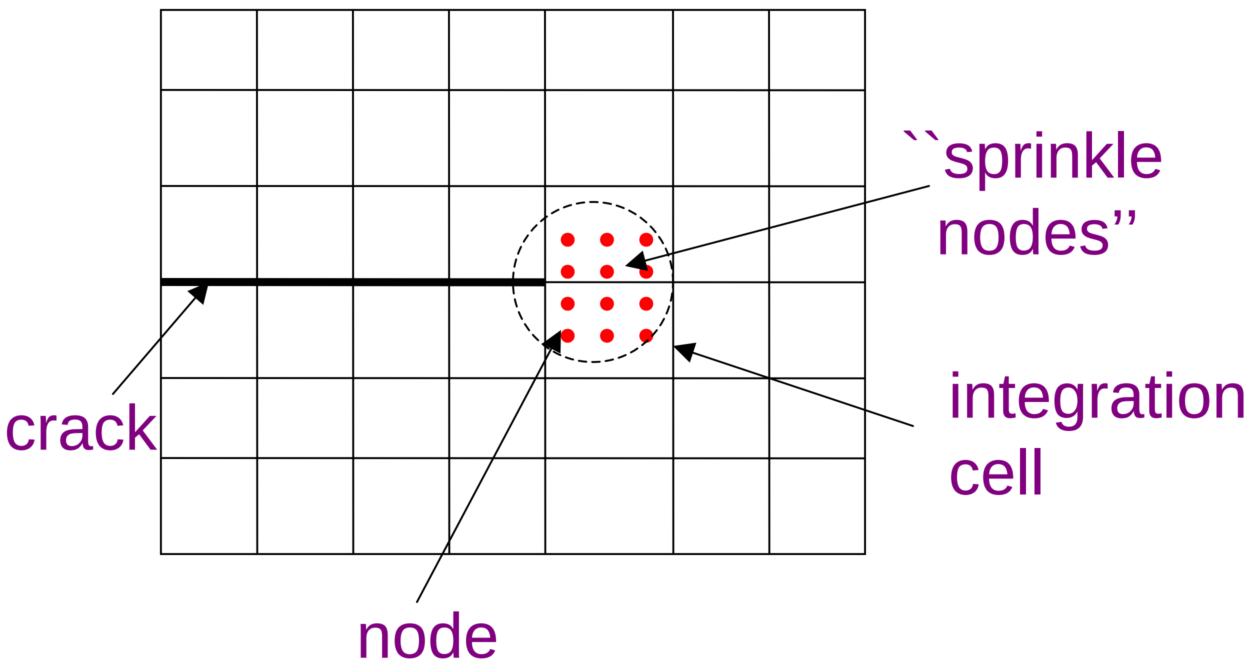
$$\mathrm{U}(\boldsymbol{\varphi}) = \sum_{i=1}^{n} \phi_i \,\|\, \mathbf{x} - \mathbf{x}_i \,\|^2$$ (second-order moment)

$$\mathrm{H}(\boldsymbol{\varphi}) = -\sum_{i=1}^{n} \phi_i \log \phi_i$$ (Shannon entropy)

subject to the **three** linear reproducing conditions

Presentation by **Marino Arroyo** forthcoming on Wednesday, July 27, 2005 (USNCCM8)

C. Nodal Refinement



Concluding Remarks

- Linked the use of the maximum entropy principle to data approximation; use of extremum principles to compute shape functions have well-established roots (Kriging, Delaunay, thin-plate splines, MLS, Laplace)

- Numerical formulation to solve the **MAXENT** problem in 1D and 2D was described, which readily extends to $\mathbf{R}^d \ (d \in \mathbf{N})$. A JAVA applet to plot meshfree shape functions has been developed

- The use of information-theoretic principles in materials and mechanics computations holds promise